Riding the Bubble with Convex Incentives

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ABSTRACT

We offer a simple portfolio choice-based rationale for the documented failure of institutional investors to trade against mispriced securities. In a dynamic model where informed and risk-averse money managers face convex (i.e., option-like) incentives, we show that the standard mean-variance component and hedging demand that make up their portfolio often imply opposite stances against mispricing. The first component represents managers’ bets against overpriced securities. By contrast, managers’ hedge against the risk of forfeiting an end-of-period “bonus” payment can over-weigh overpriced securities even with respect to the market portfolio. This “bubble-riding” component is more likely to drive the managers’ portfolio as overpricing increases. Although we do not model the informed managers’ price impact, our analysis suggests that the incentives of sophisticated investors might drive them to exacerbate security mispricing.

Keywords: Money management, convex incentives, incomplete information, mispricing.

JEL Classification: D81, D82, D83, G11, G23.

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1 Introduction

The role of institutional investors in enhancing financial markets efficiency has come under increased scrutiny following recent episodes of prolonged—perceived—mispricing. Indeed, several empirical studies document that hedge funds, as well as mutual funds and independent investment advisors, were heavily invested in technology stocks during the “tech bubble” of the late 1990s.\(^1\) Such over-investment in overvalued securities, or “bubble-riding”, has often been associated with herd behavior resulting from career concerns. Yet, some authors argue that career concerns should be substantially lessened by the use of short-term incentive contracts.\(^2\) Bubble-riding by institutional investors thus seems particularly puzzling in light of the widespread use of pay-for-performance and the high power of the incentives commonly observed in the asset management industry.

In this paper, we relate failure to trade against mispricing to the incentives of money managers. Using a dynamic investment model with no career concerns, we show that short-term contracts may not work as expected when implemented through “convex” incentives, i.e. those that reward good performance more than they penalize poor outcomes. In particular, we show that convex incentives can lead money managers to over-invest, rather than under-invest, in overpriced securities. Even in the absence of funding constraints, we argue that the resulting bubble-riding behavior can worsen precisely when mispricing heightens and be inconsistent with the behavior expected under efficient financial markets.

Convex incentives are ubiquitous in the money management industry. Among hedge funds, an explicit convexity arises from the typical fees charged to their clients. These include a flat management fee plus a “bonus” performance fee—normally, several times larger than the management fee—over profits in excess of a hurdle performance rate or a high-water mark. In the mutual fund industry, an implicit convexity results from the relation between a fund’s performance and its clients’ share purchases and redemptions. Indeed, an extensive literature (see, e.g., Chevalier and Ellison (1997), and Sirri and Tufano (1998)) documents that mutual fund inflows after good performance largely exceed outflows following poor returns. Since mutual funds receive revenue based on the value of their assets under management (AUM), such a convex flow-performance relationship

\(^1\) See, e.g., Brunnermeier and Nagel (2004), Greenwood and Nagel (2009), and Griffin, Harris, Shu, and Topaloglu (2011).

\(^2\) See, e.g., Scharfstein and Stein (1990) and Dass, Massa, and Patgiri (2008).
suggests an implicit option-like relation between mutual fund performance and managerial compensation. A further source of convexity in mutual fund managers’ incentives is created by the prevalence of bonus payments in their end-of-year compensation packages (see, e.g., Farnsworth and Taylor (2006), and Ma, Tang, and Gomez (2013)). These payments are contingent on good performance and typically represent a substantial portion of managers’ salaries.\(^3\)

We incorporate this type of non-linear incentives into the dynamic investment problem of a money manager that trades in potentially mispriced assets. Our model features informed managers alongside uninformed retail investors. Whereas only the former can assess the extent of mispricing, both types of agents are risk averse and trade continuously over a short investment horizon to maximize utility over final wealth. In particular, we assume that managers are subject to the convex incentives commonly employed in the asset management industry, with “bonus” payments that depend on performance in excess of a given benchmark.

Managers’ superior information in our model stems from their ability to observe asset fundamentals, an assumption we adopt as shortcut for high investment ability. Asset prices, in turn, are determined in equilibrium by the trading decisions of the uninformed retail investors who do not observe these fundamentals but learn about them from the observation of the asset dividends over time. Since uninformed traders’ inference of asset payoffs is subject to errors, their trading strategies can in principle push market prices away from fundamental value. Given the ensuing mispricing dynamics, our main goal is then to study the optimal trading policies of the informed money managers in partial equilibrium.

We solve for the price dynamics and for informed managers’ trading strategies in closed-form. Depending on uninformed traders’ up-to-date inferred dividend growth rate, asset prices can be higher or lower than their fundamental value, i.e. the corresponding prices in a full-information economy—as observed by the informed managers. Thus, time-varying learning by the uninformed traders leads to time-variation in the level of asset mispricing, eventually resulting in periods of large overpricing (as well as underpricing) of securities.

Under these price dynamics, we first address the question of how much an informed direct trader—one who has the same information and risk aversion as managers but faces

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\(^3\) This type of compensation packages is prevalent in the financial industry in general, beyond its pervasiveness in the asset management industry.
no convex incentives—should invest in mispriced assets. This case provides us with a standard or, following Basak, Pavlova, and Shapiro (2007), “normal” policy against which we can assess the trading of informed managers facing convex incentives. For the short investment horizon we consider in this paper, we find that such a trader’s investment policy is consistent with the expected behavior under the standard paradigm of market efficiency. In particular, (i) the informed trader over-weighs underpriced assets and under-weighs overpriced ones relative to the market portfolio, (ii) the size of the informed trader’s bets against mispricing increases in the extent of mispricing and (iii) can even result in substantial short-sale positions for largely overvalued securities.

Next, we examine the extent to which convex incentives can make money managers trade more or less aggressively against mispricing than in the absence of these incentives—the normal policy. If short-term profit-based contracts are to offset the incentives resulting from career concerns, we would expect convex incentives to induce managers to trade more aggressively than the normal policy against mispricing in our simple model without career concerns.

In contrast with this argument, we find that convex incentives can induce substantial bubble-riding behavior by money managers in our model. Specifically, the optimal dynamic trading strategy of these managers includes a mean-variance component that summarizes managers’ bets against mispricing, but also a hedging demand against the risk of underperforming or, equivalently, of forfeiting a performance-based “bonus” payment. This hedging demand features well-known “risk-shifting” and “indexing”—i.e., investing like a benchmark index—components, respectively, for managers under- and outperforming their benchmarks. Our main contribution is to show that, under mispricing, both components of informed managers’ hedging demand (i) can lead them to over-invest in overvalued assets relative to the case of no convex incentives, and (ii) can distort their investment policies more as mispricing heightens.

Although convex incentives induce similar distortions on hedge fund and mutual fund managers’ trading against mispricing, the exact mechanisms differ according to their different investment objectives. In the case of hedge funds, the hedging demand can lead managers to over-weigh the overpriced assets in their portfolios even more than the uninformed traders in our model. This behavior responds to managers’ gambles for large returns—risk-shifting—when they trail their benchmarks at an interim date, and can persist over relatively long mispricing periods. As they attain the performance fee criterion
within the period, hedge fund managers’ hedging demand seeks to lock in the interim gain by investing like their benchmark—indexing. For “absolute return” strategies—as the goals of hedge funds are commonly advertised, this benchmark resembles a (scaled) money market account. Thus, a strategy that mimics a money market account can restrict the extent to which hedge fund managers short-sell a highly overpriced stock. This self-imposed constraint on short-selling limits these managers’ bets against overvalued assets even in the absence of explicit portfolio constraints in our model. It also suggests a limited role for sophisticated investors in stabilizing financial markets in situations of large overpricing.

In the case of mutual funds, we show that the indexing component of their hedging demand can lead informed managers to trade less aggressively against both under- and overpricing than in the absence of convex incentives. Since the implicit benchmark in investors’ flows is a stock market index, the benchmark that mutual funds care about can itself be overvalued. Therefore, a hedging demand driven by the attempt to lock in an interim outperformance tends to over-weigh (respectively, under-weigh) an overvalued (undervalued) asset in their portfolios. Moreover, we show that informed mutual fund managers facing a higher sensitivity of flows to medium and bottom performance trade less aggressively against deviations in market prices from fundamental value at all mispricing levels.

We further show that both types of money managers in our model generally invest more conservatively against mispricing as their information advantage over other market participants widens. This behavior reflects managers’ increased probability of attaining outperformance as their information advantage improves—equivalently, as the expected extent of mispricing or overvaluation rises—which leads to a more conservative stance against mispricing. Following Pastor and Veronesi (2009)’s interpretation of “bubbles” as periods of heightened uncertainty, this investment pattern also provides an appealing explanation for the failure of sophisticated investors’ to trade against overpricing. These findings lead us to conclude that, when managers face convex incentives, their optimal investment strategy can in general be inconsistent with the market efficiency notion that bets against mispricing should build up as mispricing worsens.

An interesting aspect of our analysis is that we can justify some “puzzles” regarding the trading strategy of presumably sophisticated investors without recurring to behavioral arguments, and only using incentives documented in the literature—although not standard in financial models. In particular, we argue that informed hedge funds may find it optimal to
invest in overpriced stocks in a proportion even higher than the market, as documented by Brunnermeier and Nagel (2004). Second, we show how investors’ flows can induce excessive holdings of overpriced assets in mutual fund portfolios as found by Greenwood and Nagel (2009). Third, we provide an incentive-based—as opposed to financial constraint-based—explanation for the low short interest during overpricing periods as documented by, e.g., Stein and Lamont (2004). Our results are also consistent with Griffin, Harris, Shu, and Topaloglu (2011), who find that hedge funds over-invested more aggressively on tech stocks than mutual funds, which anyway invested more heavily in the same stocks than direct traders.

Our paper also contributes to the theoretical literature emphasizing rational explanations of limits to arbitrage. Allen and Gorton (1993) argue that unskilled fund managers may ride bubbles (buy overvalued assets) as a way to shift risk in response to limited liability incentives in order to pretend being skilled. Although our results draw on incentive distortions, our analysis shows that also skilled managers may choose to buy overvalued assets. Shleifer and Vishny (1997) show that managers trade less aggressively than expected against arbitrage when they face the risk of investors’ capital withdrawal before the mispricing can be completely eliminated. Abreu and Brunnermeier (2003) propose a synchronization problem behind a bubble build-up, in which sophisticated investors choose to ride the bubble in expectation that it will continue in the short run. Their analysis was generalized to a delegated portfolio management context by Sato (2009). Liu and Longstaff (2004) show that capital-constrained risk-averse arbitrageurs can trade conservatively against arbitrage opportunities and even lose money in the process. Stein (2009) suggests that sophisticated investors can ride mispricing due to an unawareness of the aggregate capital involved in eliminating the mispricing.

The bubble-riding behavior that we suggest does not hinge on financial constraints limiting sophisticated investors’ strategies but stems solely from managers’ convex incentives. Moreover, our simple asset pricing setup leaves out synchronization and “crowded-trade” risks leading to over-investment of overpriced assets. In this sense, the effects driving bubble-riding in our model are closer to those analyzed by DeMarzo, Kaniel, and Kremer (2008). These authors show how relative concerns can lead to the emergence of bubbles in an equilibrium rational asset pricing model. Our partial equilibrium analysis also stresses the role of relative performance concerns in the behavior of bubble-riders, and shows that these concerns can be exacerbated in the presence on convexities in incentives. Overall, we
think that our explanation can complement the existing rationalizations of bubble-riding behavior using only the type of compensation arrangements for money managers commonly observed in practice.

The paper is structured as follows. In Section 2 we describe the economic setting. We derive price dynamics and the optimal trading strategies of informed money managers in response to these prices in Section 3. In Section 4 we simulate our model under alternative incentive structures and mispricing scenarios. We close the paper with conclusions in Section 5.

2 Economic Setting

We are interested in the effects of convex compensation on the incentives of informed—i.e., sophisticated—institutional investors to trade against security mispricing over short-term periods. We concentrate our analysis on the behavior of hedge funds and mutual funds, for which explicit or implicit option-like compensation structures have been extensively reported in the literature. For analytical convenience, we model an economy populated by only a few informed managers along with a large number of individual retail investors. We thus adopt a partial equilibrium approach in which prices are determined in equilibrium by the trading strategies of the retail investors with no impact from the informed managers’ investment policies. Price determination by the retail traders potentially gives rise to asset overvaluation (“bubble-like” prices) and, more generally mispricing, in our model. At the same time, it allows us to derive the investment policies of managers in response to this mispricing in closed-form. Regardless, we expect our results to have implications for the more realistic case in which informed managers can affect equilibrium prices.

2.1 Financial Markets

We consider a pure exchange economy over the finite period $t \in [0, T']$. Financial markets consist of one risk-less asset $\beta$ paying constant interest rate $r$ per unit of time, and one risky

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4 See references in the introduction.

5 Our approach to the informed managers’ trading under mispricing is in the spirit of DeLong, Shleifer, Summers, and Waldman (1990), who analyze the survival of irrational traders in a model in which noise traders do not affect prices. By contrast, we focus on the trading of informed managers in partial equilibrium when the less informed traders determine prices in general equilibrium.
asset \( S \) (henceforth, a “stock”) which represents a claim to the dividend \( D_{T'} \) at \( t = T' \). \( D_{T'} \) is the terminal value of a dividend process with initial value \( d_0 \) and dynamics given by:

\[
dD_t = D_t (\rho dt + \delta dB_t),
\]

where the dividend’s mean growth rate \( \rho \) and volatility \( \delta \) are positive constants, and \( B \) is a standard Brownian motion process under the probability measure \( P \) that explains the dynamics of this economy. Everyone observes \( \delta \); however, as we describe later, only money managers with superior information observe \( \rho \). The constant \( \rho \) is the unobserved realization at \( t = 0 \) of a random variable with normal distribution \( N(\rho_0, v_0) \), for given constant prior \( \rho_0 \) and variance \( v_0 \geq 0 \).

We assume that the risk-less asset is in zero net supply, while the stock is in unit supply. The stock price satisfies the following dynamics:

\[
dS_t = S_t (\mu_t dt + \sigma_t dB_t),
\]

with mean rate of return \( \mu_t \) and volatility \( \sigma_t > 0 \) to be determined in equilibrium by the strategies of uninformed retail investors.

### 2.2 Retail Investors

The economy is populated by a continuum of retail investors, each of which starts with an endowment of one share of the stock. They all have identical constant relative risk aversion (CRRA) preferences, with coefficient \( \gamma > 1 \). Retail investors are uninformed, in the sense that they do not observe the realized value of the dividend growth rate \( \rho \) at \( t = 0 \). They belong to either one of two groups:

- **A mass one of direct** investors that trade in financial markets on their own accounts. We refer to these as \( U \)-investors. At each time \( t \in [0, T'] \), \( U \)-investors allocate a fraction \( \phi^U_t \) of their wealth \( W^U_t \) to the stock and the remaining fraction to the risk-less asset to maximize utility from final wealth at time \( T' \). Given initial wealth \( w_0 = S_0 \) their wealth process evolves according to:

\[
dW^U_t = W^U_t (r + \phi^U_t (\mu_t - r)) dt + W^U_t \phi^U_t \sigma_t dB_t.
\]
Since they do not observe $\rho$ at $t = 0$, $U$-investors have to infer it from the observation of dividends during the investment period. We explain the dynamics of the posterior of the expected dividend growth rate in section 3.1.

- A mass zero of indirect ($I$-) investors that delegate their wealth to professional money managers $M$ at the beginning of the period. The zero-mass assumption for $I$-investors follows from our partial equilibrium approach to the informed managers’ problem. We take these investors’ delegation decision as exogenously given and potentially justified by the information advantage that money managers have over all retail investors.

### 2.3 Money Managers

All money managers are identical and have no initial endowment. Instead, they invest $I$-investors’ delegated wealth in exchange for a compensation $f_TW_T$, which is the product of a fee rate $f_T$ and of assets under management (AUM) $W$ at the (possibly interim) date $T \leq T'$. Our main analysis focuses on the particular case in which the compensation date $T$ coincides with the investment horizon $T'$, i.e. $T = T'$. However, our analysis does not depend on this specific assumption and we also examine the robustness of our results to the more general case $T < T'$. For consistency, we then keep the separate notation for the interim compensation date $T$ and the final horizon $T'$ and distinguish the cases $T = T'$ and $T < T'$ whenever it is appropriate hereafter.

Similarly to $U$-investors, managers maximize utility from final wealth with identical relative risk aversion $\gamma$. Managers’ final wealth consists of the compensation $f_TW_T$ in the case $T = T'$, and is proportional to the terminal AUM with time-$T$ value $f_TW_T$ in the case $T < T'$. Each portfolio manager dynamically chooses an investment policy $\phi_t$ representing the fraction of $W_t$ that is allocated to the stock at time $t \in [0, T']$. Given $W_0 = w_0$, the value of the portfolio (AUM) follows:

$$dW_t = W_t(r + \phi_t(\mu_t - r))dt + W_t\phi_t\sigma_tdB_t.$$  \hspace{1cm} (4)

The compensation fee $f_T$ is a function of funds’ performance relative to a benchmark

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6 The proportionality constant in this case depends on the particular type of money manager (hedge funds or mutual funds) that we examine, as explained below. In any case, it does not change incentives in the margin nor, therefore, the managers’ optimization problem.
index \( Y \) (henceforth just “benchmark”). For an arbitrary initial value \( Y_0 \), this benchmark represents a long-only fixed-weight portfolio investing a fraction \( \phi^Y \in [0, 1] \) of its value in the stock and the remaining fraction in the risk-free asset.\(^7\)

\[
dY_t = Y_t \left( r + \phi^Y (\mu_t - r) \right) dt + Y_t \phi^Y \sigma_t dB_t.
\]  

(5)

The fee rate \( f_T \) is specified as follows:

\[
f_T = k \left( \frac{R_T}{\zeta R_T^Y} \right)^{\alpha_1} \mathbb{1}_{\{R_T < \tilde{\zeta} R_T^Y\}} + k \left( \frac{R_T}{\tilde{\zeta} R_T^Y} \right)^{\alpha_2} \mathbb{1}_{\{R_T \geq \tilde{\zeta} R_T^Y\}},
\]

(6)

where \( k, \tilde{\zeta} > 0, 0 \leq \alpha_1 \leq \alpha_2, R_T \equiv W_T/W_0, \) and \( R_T^Y \equiv Y_T/Y_0 \). This specification is a generalization of the function used to represent mutual fund investors’ flow-performance relationship by Basak and Makarov (2014), and is similar to the incentive function in Kaniel and Kondor (2013).

The fee rate (6) is positive and (weakly) increasing in a fund’s performance relative to the benchmark \( R_T/R_T^Y \). More important for our purpose, it implies an asymmetric relation between relative performance and perceived fees whenever the slope \( \alpha_1 \) in the underperformance region, \( \{R_T < \tilde{\zeta} R_T^Y\} \), is smaller than the slope \( \alpha_2 \) in the outperformance region, \( \{R_T \geq \tilde{\zeta} R_T^Y\} \). In particular, \( \alpha_1 < \alpha_2 \) implies that the fees perceived by managers increases with performance at a faster rate when relative performance \( R_T/R_T^Y \) exceeds a threshold \( \tilde{\zeta} \). This asymmetric relation implies an “option-like” compensation scheme for money managers, according to which managers are rewarded a “bonus” payment given by the difference in fee rates between the outperformance and underperformance regions in (6) when they perform relatively well.

We adopt alternative parameterizations of the fee rate to reflect the compensation structure for two type of institutional investors:

**Hedge funds:** If the informed investors in our setup are hedge fund managers, we assume they are compensated through a base management fee rate that is proportional to AUM, plus a substantially higher incentive fee rate that is proportional to the realized

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\(^7\) Setting the benchmark’s initial value to equal \( Y_0 = S_0^{\phi^Y} \beta_0^{1-\phi^Y} e^{-\frac{\phi^Y}{2} (1-\phi^Y)(\delta^2 + \nu T')} T' \), this benchmark is equivalent to a claim to a Cobb-Douglas composite basket \( Y_T' = S_T^{\phi^Y} \beta_T^{1-\phi^Y} \) to be paid at the end of the period.
profits in excess of a preassigned benchmark. This specification is meant to reflect the typical practice in the hedge fund industry of charging a 1%-2% management fee along with an incentive fee equal to 20% of investment profits beyond a stipulated benchmark performance. Given that most hedge funds’ preassigned benchmark is not a market index but a money market rate such as LIBOR plus a spread, we further assume that the benchmark is the risk-free rate $r$ plus (possibly) a positive hurdle rate $h$. This implies $\phi^Y = 0$ in (5), in which case $Y_T = \beta_0 e^{rT}$. Defining the continuously-compounded rates $r_T \equiv \ln (R_T) / T, r_Y^T \equiv \ln (R_Y^T) / T = r$, and setting the threshold $\zeta = e^{hT}$ for the spread (hurdle rate) $h \geq 0$, for any $\alpha > 0$ we can write:

$$ k \left( \frac{R_T}{\zeta R_Y^T} \right)^\alpha = k e^{\alpha(r_T-(r+h))T}. \quad (7) $$

A first-order approximation of the RHS of (7) around $r_T = r + h$ gives:

$$ k \left( \frac{R_T}{\zeta R_Y^T} \right)^\alpha \approx kT + k\alpha(r_T - (r + h))T, \quad (8) $$

Applying (8) to the two terms in the RHS of (6) and setting $\alpha_1 = 0$ implies a fee rate:

$$ f_T \approx kT + kT \alpha_2 (r_T - (r + h))^+, \quad (9) $$

where $x^+ \equiv \max(0, x)$. Equation (9) makes it clear how we can parameterize the fee rate (6) to approximate the typical fee structure for hedge funds, consisting of a management fee rate $kT$ plus an option-like incentive fee rate $kT \alpha_2$ on fund profits in excess of the hurdle performance $(r + h)$. We assume hedge funds are liquidated and paid back to their owners at $t = T$, after deducting managers’ compensation $f_T W_T$. No additional fund share purchases or redemptions by hedge funds’ $I$-investors occur during $[0, T]$. Whenever we set $T < T'$, we assume that hedge fund managers trade for their own account—consisting of

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8 This is consistent with hedge funds’ goal of delivering “absolute returns” in all market conditions. The fee structure often includes a “high-water mark” stating that following a year in which the fund declined in value, the fund would first have to recover those losses before any incentive fee can be charged. Such provisions may reduce the long-term risk-seeking incentives of a hedge fund manager, as analyzed by Hodder and Jackwerth (2007), Panageas and Westerfield (2009), and Drechsler (2013). Since we focus on the trading behavior of managers over short-horizons, we follow Buraschi, Kosowski, and Sritrakul (2011) in assuming that the high-water mark is prespecified at the beginning of the period, and allow for differences in high-water marks by varying our spread parameter $\zeta$ in (6).
the fees collected at \( t = T \)—during \([T, T']\).

**Mutual funds:** If the informed investors in our setup are mutual fund managers, we assume they charge a base management fee rate \( m \) in proportion to AUM. At \( t = T \), but at no other \( 0 \leq t < T \), mutual funds’ I-investors purchase or redeem additional fund shares depending on managers’ performance during \([0, T]\) relative to the benchmark \( Y \) according to an exogenously given flow-to-relative performance relationship (F-PR) \( q_T \):

\[
q_T = q \left( \frac{R_T}{\zeta_R Y_T} \right)^{\alpha_1} \mathbb{1}\{R_T < \zeta_R Y_T\} + q \left( \frac{R_T}{\zeta_R Y_T} \right)^{\alpha_2} \mathbb{1}\{R_T \geq \zeta_R Y_T\},
\]

(10)

with \( q > 0 \). Defining \( k \equiv mq \), we see that mutual fund managers’ fee rate \( f_T = mq_T \) follows the specification (6). This functional form allows flows to be sensitive (and potentially concave, if \( \alpha_1 < 1 \)) to medium and low relative performance. At the same time, \( f_T \) reflects the well-documented convexity in the sensitivity of flows to performance (see, e.g., Chevalier and Ellison (1997), and Sirri and Tufano (1998)) for \( \alpha_1 < \alpha_2 \), according to which outperforming funds receive a disproportionally high amount of inflows.\(^9\) The F-PR (6) can also capture linear relationships (\( \alpha_1 = \alpha_2 = 1 \)), near-linear relationship (\( \alpha_1 = \alpha_2 \neq 1 \)), as well as no relationship at all (\( \alpha_1 = \alpha_2 = 0 \)). Although we present our main results for general benchmarks, we specialize most of our numerical analysis to the case of all-equity mutual funds for which \( \phi^Y = 1 \). No additional fund share purchases or redemptions by I-investors take place until \( t = T' \), when funds are liquidated and paid back to their owners, after deducting the final managerial fee compensation \( mW_{T'} \).

For both types of managers, the sequence of events during \([0, T']\), for \( T \leq T' \), is summarized in the timeline of Figure 1. Crucially, hedge fund or mutual fund managers are more informed than U-investors in our setup, making delegation potentially valuable to the I-investors. In particular, managers observe the realization of the dividend growth rate \( \rho \) at \( t = 0 \), whereas U-investors learn its value from their prior and the observation of the dividend process \( D_t \) over the period \( t \in [0, T'] \).

\(^9\) While many empirical studies document no sensitivity of flows to poor relative past performance (e.g. Sirri and Tufano (1998)), many others (e.g. Huang, Wei, and Yan (2007)) find it is positive, although lower than the sensitivity to medium or high relative returns.
3 Equilibrium

We are ultimately interested in understanding trading under mispricing, i.e. the trading by hedge fund or mutual fund managers—possibly facing convex incentives—who observe that asset prices do not reflect all the information available to them. This requires solving for the dynamics of prices in a first stage, and for the optimal portfolio of informed investors under these prices in a second stage. All proofs are given in Appendix A.

Equilibrium in our model consists of a set of trading strategies and asset prices such that: (i) $U$-investors’ and managers’ individual investment policies are optimal, and (ii) bond and stock markets clear.\footnote{Note that we do not require $I$-investors’ delegation decision to be optimal, which we assume as exogenously given in our model. However, the assumed information advantage of managers over $U$-investors ensures that, for the parameterizations that we consider later in the analysis of our model, there exists a threshold level of uncertainty $\bar{v}$ above which $I$-investors are at no disadvantage relative to $U$-investors. More precisely, for all prior uncertainty $v_0 > \bar{v}$, $I$-investors can attain—at least—as high an expected certainty equivalent returns from delegation, after low enough fees, as from investing based on their incomplete information for their own accounts.}

Assuming bonds are in zero net supply, market clearing under our simplifying zero-mass assumption for $I$-investors along with the requirement of no-arbitrage implies:\footnote{The condition of absence of arbitrage in the stock market requires that the stock price equals the liquidation dividend at the terminal date: $S_{T'} = D_{T'}$.}

\begin{equation}
W^U_T = S_{T'} = D_{T'}.
\end{equation}

Based on equation (11), we next provide closed-form expressions for the asset prices in this economy by solving for the optimal strategies of $U$-investors.
3.1 Price Dynamics

$U$-investors start the investment period with a prior distribution for $\rho$ and update it over time according to Bayes rule, based on the arrival of information $D_t$. We denote by $\tilde{P}$ the probability space that describes the dynamics of the dividend process according to $U$-investors’ priors. Letting $\tilde{E}$ denote the expectation under $\tilde{P}$, the distribution of $\rho$ conditional on $D_t$ is Gaussian, with mean $\tilde{\rho}_t \equiv \tilde{E}[\rho|D_t]$ and variance $v_t \equiv \tilde{E}[(\rho - \tilde{\rho}_t)^2|D_t]$ satisfying:

\begin{align}
\begin{cases}
    d\tilde{\rho}_t = \frac{v_t}{\delta} \left( \frac{dD_t}{D_t} - \tilde{\rho}_t dt \right) \equiv \frac{v_t}{\delta} d\tilde{B}_t, \\
    dv_t = -\frac{v_t^2}{2\delta} dt,
\end{cases}
\end{align}

(12)

for the initial values $\tilde{\rho}_0 = \rho_0$ and $v_0$. $\tilde{B}_t$ is a standard Brownian motion with respect to the filtration $\mathcal{F}_t^D$ generated by the dividend process $D$ under $\tilde{P}$, with dynamics $d\tilde{B}_t \equiv \frac{1}{\delta} \left( \frac{dD_t}{D_t} - \tilde{\rho}_t dt \right) = dB_t + \frac{\rho - \tilde{\rho}_t}{\delta} dt$. Letting $\tilde{\mu}_t \equiv \tilde{E}[\mu_t|D_t]$ and $\tilde{\eta}_t \equiv \frac{\tilde{\mu}_t - r}{\sigma_t}$ be $U$-investors’ time-$t$ inferred stock mean rate of return and market price of risk under $\tilde{P}$, $U$-investors allocate $\phi_t^U$ to the stock market to maximize expected utility over terminal wealth:

$$\max_{(\phi_t^U)_{t \in [0,T]}} \tilde{E}_0 \left[ \frac{(W_{T}^{U})^{1-\gamma}}{1 - \gamma} \right],$$

(13)

subject to initial wealth $w_0$ and the re-stated, in terms of observable variables, self-financing constraint (3):

$$dW_t^U = W_t^U (r + \phi_t^U (\tilde{\mu}_t - r)) dt + W_t^U \phi_t^U \sigma_t d\tilde{B}_t.$$  

(14)

From $U$-investors’ perspective, markets are complete with respect to the observable states of the economy (a single risky asset $S$ driven by a single Brownian motion $\tilde{B}$). We can then solve problem (13) as in an equivalent full-information framework. Absent arbitrage opportunities, $U$-investors see financial markets as driven by a unique state-price deflator (SPD) $\tilde{\pi}$ with dynamics $d\tilde{\pi}_t = -r \tilde{\pi}_t dt - \tilde{\pi}_t \tilde{\eta}_t d\tilde{B}_t$. The dynamic budget constraint (14) can be restated (see e.g. Karatzas and Shreve (1998)) as:

$$\tilde{E}_0 [\tilde{\pi}_T W_{T}^{U}] = w_0.$$  

(15)

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12 See, e.g., Liptser and Shirayayev (2001).
Using the martingale/duality approach of Cox and Huang (1989) and Karatzas, Lehoczky, and Shreve (1987), the dynamic optimization problem (13) can be solved as a static problem over final payoffs $W_{U'}$. Standard individual optimization results along with the market clearing condition (11) then lead to the following

**Proposition 1.** Let $\tau' = T' - t$. Equilibrium stock prices and uninformed investors’ SPD are given by:

$$S_t = D_t \exp \left\{ \left( \tilde{\rho}_t - r - \gamma \delta^2 - \left( \gamma - \frac{1}{2} \right) v_t \tau' \right) \tau' \right\},$$

(16)

$$\tilde{\pi}_t = \lambda^{-1} D_t^{-\gamma} \exp \left\{ \left( r - \gamma \tilde{\rho}_t + \frac{1 + \gamma}{2} \gamma \delta^2 + \frac{\gamma^2}{2} v_t \tau' \right) \tau' \right\},$$

(17)

where $\lambda > 0$ is the Lagrange multiplier of the equivalent static problem and its solution is given in Appendix A. Under the probability $\tilde{P}$, equilibrium stock mean return, volatility and market price of risk are time-varying and deterministic, as given by:

$$\tilde{\mu}_t = r + \gamma \sigma_t^2,$$

$$\sigma_t = \delta + \frac{v_t}{\delta} \tau',$$

$$\tilde{\eta}_t = \gamma \sigma_t.$$

(18)

The stock price dynamics in equations (16) are affected by investors’ estimate of the dividend growth rate $\tilde{\rho}_t$ and by their uncertainty $v_t$. $U$-investors’ incomplete information can then introduce a wedge between the stock market price in this economy and its fundamental value, understood as the equilibrium stock price $S^{CI}$ that would prevail if all traders in the economy had complete and symmetric information about the dividend growth rate $\rho$ (i.e., $\rho_0 = \rho$ and $v_0 = 0$).

Arguably, any situation in which $S/S^{CI} \neq 1$ would be perceived as stock mispricing by fully informed investors such as the money managers in our setup. Hence, we measure the extent of stock overvaluation as of time $t < T'$ by the quantity $OV_t \equiv (S_t/S^{CI}_t)^{1/\tau'} - 1$, and the extent of mispricing by the quantity $MP_t \equiv |OV_t|$. We say that stock mispricing reflects overvaluation or overpricing (respectively, undervaluation or underpricing) whenever $OV_t > 0$ ($OV_t < 0$). Since by no-arbitrage $S_{T'} = D_{T'} = S^{CI}_{T'}$, stock mispricing equals

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13 As can be seen from the proof of Corollary 1 below, the ratio $S_t/S^{CI}_t$ depends on the time to maturity $\tau'$. The definitions of $OV_t$ and $MP_t$ then make the extent of mispricing at different dates comparable.
zero at the terminal date $T'$. The following Lemma characterizes the stock fundamental value and the equilibrium mispricing and overvaluation as perceived by informed managers at any interim period:

**Corollary 1.** Let $\tau' = T' - t > 0$. Under complete information for all agents in the economy, equilibrium stock prices are given by:

$$S_t^{CI} = D_t \exp \left\{ \left( \rho - r - \gamma \delta^2 \right) \tau' \right\}.$$  \hfill (19)

Equilibrium stock mean return, volatility and market price of risk are given by:

$$\mu^{CI} = r + \gamma \left( \sigma^{CI} \right)^2,$$

$$\sigma^{CI} = \delta,$$

$$\eta^{CI} = \gamma \sigma^{CI}.$$  \hfill (20)

The time-$t$ stock overvaluation $OV_t$, as perceived by fully-informed agents, is given by:

$$OV_t = \exp \left\{ \left( \hat{\rho}_t - \rho - \left( \gamma - \frac{1}{2} \right) v_t \tau' \right) \right\} - 1.$$  \hfill (21)

As is intuitive, we see from equation (21) that an over-estimation of the mean dividend growth rate by $U$-investors, $\hat{\rho}_t > \rho$, will typically lead to stock overpricing.\footnote{The stock will be overpriced as long as $U$-investors over-estimate the dividend growth rate by a large enough margin: $\hat{\rho}_t - \rho > (\gamma - .5)v_t \tau' > 0$. This implies that a low enough over-estimation of fundamentals, $(\gamma - .5)v_t \tau' > \hat{\rho}_t - \rho > 0$, is still consistent with an underpriced stock in our economy with sufficiently risk-averse $U$-traders.} Moreover, the extent of overpricing $OV_t$ is increasing in the over-estimation margin $\hat{\rho}_t - \rho$, consistent with the intuition that better perceived dividend growth prospects leads $U$-investors to push prices up.

We further note that, due to the equilibrium restriction (11), $U$-investors are fully
invested in the stock at all times, i.e. \( \phi_U' = \phi^U = 1 \) for all \( t \in [0, T'] \). This naturally leads to the interpretation of the stock price \( S \) (equivalently, \( U \)-investors’ wealth \( W_U \)) as the value of the market portfolio in our model. We will draw on this interpretation later on as we assess the trading policies of the informed investors in this economy. We expand on relation between over-estimation of fundamentals, stock overpricing and informed investors’ investment decisions in the next section.

### 3.2 Optimal Investment Strategy of Informed Money Managers

Informed managers observe the true value of the dividend growth rate \( \rho \). They see the stock price dynamics under the true probability \( P \) as given by:

\[
dS_t = S_t \left( \tilde{\mu}_t dt + \sigma_t \frac{\rho - \tilde{\rho}_t}{\delta} dt + dB_t \right)
\]

\[
= S_t \left( \left( \tilde{\mu}_t + \sigma_t \frac{\rho - \tilde{\rho}_t}{\delta} \right) dt + \sigma_t dB_t \right),
\]

with the dynamics of \( U \)-investors’ estimated growth rate \( \tilde{\rho}_t \) as given by

\[
d\tilde{\rho}_t = \frac{v_t}{\delta^2} (\rho - \tilde{\rho}_t) dt + \frac{v_t}{\delta} dB_t.
\]

Under the true probability \( P \), the stock mean return \( \mu_t \equiv \tilde{\mu}_t + \sigma_t \frac{\rho - \tilde{\rho}_t}{\delta} \) is stochastic and follows a mean-reverting process: higher inferred dividend growth by \( U \)-investors leads directly to lower mean stock returns today (via the contemporaneous negative impact of \( \tilde{\rho}_t \) on \( \mu_t \)), but indirectly to higher expected returns over the next instant \( t + dt \) (via the negative impact of \( \tilde{\rho}_t \) on \( d\tilde{\rho}_t \) in (23)). The market price of risk under \( P \) is also stochastic and consists of the sum of the market price of risk under \( \tilde{P} \) and a term proportional to the uninformed investors’ estimation error \( \rho - \tilde{\rho}_t \): \( \eta_t = \tilde{\eta}_t + \frac{\rho - \tilde{\rho}_t}{\delta} \).

Markets are complete for money managers, so they see financial markets as driven by

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\(^{15}\) In fact, the market price of risk under the true probability \( P \) follows a first-order autoregressive process, as given by:

\[
d\eta_t = -\frac{v_t}{\delta^2} (\eta_t dt + dB_t),
\]

for \( 0 < v_t/\delta^2 < 1 \).
a unique SPD $\pi$ with dynamics $d\pi_t = -r\pi_t dt - \pi_t \eta_t dB_t$, i.e.:

$$\pi_t = e^{-rt} \int_0^t \eta_s^2 ds - \int_0^t \eta_s dB_s = \pi_t \xi_t,$$

where $\xi_t \equiv \exp \left\{ -\frac{1}{2}\int_0^t \left( \frac{\tilde{\rho} - \rho}{\delta} \right)^2 ds - \int_0^t \frac{\tilde{\rho} - \rho}{\delta} dB_s \right\}$ is the likelihood process (a $P$-martingale) for the measure transformation from $P$ to $\tilde{P}$: $\xi_t = \frac{d\tilde{P}}{dP}$ on $\mathcal{F}_t$. Therefore, the extent to which managers’ SPD, $\pi_t$, differs from direct investors’, $\tilde{\pi}_t$, depends on the size and sign of the estimation error $\rho - \tilde{\rho}_t$.

According to equation (21), informed managers observe that time-varying learning by $U$-investors induces time-variation in the level of asset mispricing, potentially resulting in sustained periods of stock overpricing or underpricing. Under these price dynamics, we first address the question of how much an informed direct trader—one who has the same information and risk aversion as the manager but does not face convex incentives—should invest in the mispriced stock.

### 3.2.1 Benchmark Case: Investment Policy without Convex Incentives

In order to single out the effects of convex incentives on managers’ trading strategies, we first examine the dynamic investment policy that managers would follow if they were trading on their own account. Under these conditions, managers would allocate the same weight in the stock as retail investors with the same information advantage and risk aversion would. Because such policy would entail no incentive misalignment between managers and delegating ($I$-) investors, we follow Basak, Pavlova, and Shapiro (2007) in referring to this standard (default) policy as the normal ($N$) policy.

For any $\tilde{\gamma} > 1$, we define $\phi_{\tilde{\gamma},t}^N$ as the time-$t$ ($t \in [0, T']$) optimal normal trading in the stock of an investor with RRA coefficient $\tilde{\gamma}$. Proposition 2 characterizes $\phi_{\tilde{\gamma},t}^N$ along with the associated portfolio value process $W_t^N$:

**Proposition 2.** For $t \in [0, T']$, the normal trading strategy $\phi_{\tilde{\gamma},t}^N$ and associated portfolio value process $W_t^N$ are given by:

$$\phi_{\tilde{\gamma},t}^N \equiv \frac{\delta^2 + v_t \tau' \eta_t}{\delta^2 + \frac{v_t}{\tilde{\gamma} \tau'} \tilde{\gamma} \sigma_t}.$$

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where for any $\psi \in (0, 1)$ and $0 \leq t \leq t' \leq T'$

$$Z_{\psi, t, t'} \equiv \delta^\psi \sqrt{\frac{(\delta^2 + \psi v_t(t' - t))^{1-\psi}}{\delta^2 + (1 - \psi)v_t(t' - t)}} \exp \left\{ -\psi \rho(t' - t) - \frac{\psi(1 - \psi)\delta^2(t' - t)}{\delta^2 + (1 - \psi)v_t(t' - t)} \frac{\eta_t^2}{2} \right\},$$

and $\lambda_N = \left( \frac{Z_1 - \tilde{\gamma} \rho}{w_0} \right)^{\frac{\tilde{\gamma}}{2}}$.

Comparing the portfolio weight in the stock of the $U$-investors (the market portfolio), $\phi_{\gamma, t}^U$, to the normal policy of an equally risk-averse investor, $\phi_{\gamma, t}^N$, we have the following

**Corollary 2.** For $t \in [0, T']$, the normal excess holding of the stock relative to the market is given by:

$$\phi_{\gamma, t}^N - \phi_{\gamma, t}^U = -\frac{1}{\gamma} \tilde{\rho}_t - \rho - (\gamma - 1)v_t\tau'.$$

(28)

Thus, the normal portfolio implies a lower stock holding than the market, $\phi_{\gamma, t}^N < \phi_{\gamma, t}^U$, iff:

$$\tilde{\rho}_t > \rho + (\gamma - 1)v_t\tau'$$

$$\equiv O V_t > \exp \left\{ -\frac{1}{2} v_t\tau' \right\} - 1.$$

(29)

The normal portfolio implies a higher stock holding than the market if the converse of (29) holds; both holdings are the same when (29) holds as an equality.

Table 1 summarizes the relationship between over-estimation of fundamentals, stock overpricing and the normal policy. Except for a typically low-probability range of underpricing, $OV_t \in (\exp \left\{ -\frac{1}{2} v_t\tau' \right\} - 1, 0)$ (corresponding to an estimation error $(\tilde{\rho}_t - \rho) \in ((\gamma - 1)v_t\tau', (\gamma - 0.5)v_t\tau'))$, we see that for an overpriced stock, $S_t > S_t^{CJ}$, the normal portfolio underweights the stock relative to the market $(\phi_{\gamma, t}^N < \phi_{\gamma, t}^U)$, and conversely for an

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16 In these states, the normal portfolio under-weighs slightly underpriced securities. This occurs because positive but small enough over-estimation of the dividend growth rate by $U$-investors, $0 \leq (\gamma - 1)v_t\tau' < \tilde{\rho}_t - \rho < (\gamma - 0.5)v_t\tau'$, does not translates into stock overpricing (see Section 3.1) but still leads to lower-than-normal stock holdings. However, these states occur with low probability for short enough investment horizons $T'$. 

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underpriced security \((\phi_{N,\gamma,t} > \phi^U\) for \(S_t < S^C_t\)). Moreover, rewriting equation (28) as:

\[
\phi_{N,\gamma,t}^N - \phi^U = -\frac{1}{\gamma} \ln(1 + OV_t) + \frac{1}{2} v_t \tau'_t,
\]

we see that a higher overvaluation leads to larger stock under-weighting in the normal portfolio relative to the market, potentially resulting in sizable short positions in the stock when overpricing is high enough.

<table>
<thead>
<tr>
<th>(\hat{\rho}_t - \rho)</th>
<th>((-\infty, 0])</th>
<th>((0, (\gamma - 1)v_t \tau'_t])</th>
<th>(((\gamma - 1)v_t \tau'_t, (\gamma - 0.5)v_t \tau'_t])</th>
<th>(((\gamma - 0.5)v_t \tau'_t, +\infty])</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\text{sgn}(OV_t))</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>+</td>
</tr>
<tr>
<td>(\text{sgn}(\phi_{N,\gamma,t}^N - \phi^U))</td>
<td>+</td>
<td>+</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

Table 1: Relation Between Fundamentals, Overpricing and the Normal Portfolio

Summing up, for the short investment horizon we consider in this paper we have shown that the normal policy is consistent with the commonly expected behavior of an informed trader under efficient financial markets. In particular, (i) the informed trader over-weighs underpriced assets and under-weighs overpriced ones relative to the market portfolio, (ii) the size of the informed trader’s bets against mispricing increases in the extent of mispricing and (iii) can even result in substantial short-sale positions for largely overvalued securities. In the next section we contrast this with the behavior of informed traders under convex incentives—money managers.

### 3.2.2 Informed Money Managers

Previous authors have suggested high-powered incentives as contracting features that can alleviate “bubble-riding” behavior associated with career concerns in money management (see, e.g., Scharfstein and Stein (1990), Dass, Massa, and Patgiri (2008)). The argument is based on the intuition that a high enough weight on managers’ short-term performance should offset the negative effects of a loss of reputation, making managers more willing to deviate from the herd and away from overvalued assets. In this section and the next we assess the validity of this argument under the convex managerial incentives that we describe in section 2.
Within our setup, a simple low-powered compensation arrangement consists in setting the fee rate equal to an arbitrarily small positive constant \( c \), i.e. \( f_T = c \). Under this arrangement, it is straightforward to show that managers’ optimal trading strategy equals the normal policy at all times. If high-powered compensation arrangements such as option-like performance fees are to offset potential incentives to over-invest in overpriced assets in a more general setup, we would expect the investment policy under performance fees to induce at least as an aggressive trading against mispricing as the normal policy.

We start by analyzing whether this is indeed the case in the simplest possible setup with no career concerns for the money managers in our model. This corresponds to the one-period investment problem in which managers collect their compensation fees only at the end of the investment horizon: \( T = T' \).

**One-Period Investment Problem** \((T = T')\): This case allows us to isolate the effects of profit-based incentives on managers’ trading against mispricing, beyond any effect potentially induced by career concerns. For this reason, we present most of our results in relation to this case, and assess the robustness of these results to the case \( T < T' \) later on in the paper. Whenever possible, we relate managers’ optimal investment policy to the portfolios \( \phi^Y \), \( \phi^U \) and \( \phi^N_{\tilde{\gamma}} \) corresponding to the allocation in the stock of the benchmark portfolio, the \( U \)-investors’ portfolio and the normal policy (for a given RRA coefficient \( \tilde{\gamma} \)), as introduced in previous sections.

We introduce the following notation: let \( \tau' = T' - t \) denote the time remaining to horizon and \( \zeta = \tilde{\zeta}W_0/Y_0 \) the normalized performance threshold. In addition, let \( \gamma_i \equiv \gamma + \alpha_i(\gamma - 1) \), for \( i = 1, 2 \), denote managers’ effective relative risk aversion in the underperformance and outperformance regions of section 2.3. This interpretation of \( \gamma_i \) follows from computing the RRA coefficient corresponding to managers’ utility function when wealth is augmented by the fee rate (6) and implies that, for \( \alpha_2 > \alpha_1 > 0 \), managers’ effective relative risk aversion in the outperformance region increases by \( \gamma_2 - \gamma_1 > 0 \) relative to the underperformance region. This is the consequence of the higher sensitivity of expected fees to relative performance when their funds do better than their benchmarks.\(^{17}\) Our main result is stated in

\(^{17}\) To see this, note that changes in actual wealth are augmented by a fee rate \( k \left( W_T/\zeta W_T^Y \right)^{\alpha_2} \) in the outperformance region, but only by a flow rate \( k \left( W_T/\zeta W_T^U \right)^{\alpha_1} \) outside of it. The fee charged by a top performer is increasing in wealth at a higher rate. Therefore, effective wealth fluctuates more in response to the same change in actual wealth in this region than in the underperformance region, raising manager’s effective risk aversion.
the following:

**Proposition 3.** When \( T = T' \), the informed managers’ optimal trading in the stock during \([0,T']\) is given by:

\[
\hat{\phi}_t = \omega_t \phi_{\gamma_1,t}^N + (1 - \omega_t) \phi_{\gamma_2,t}^N + \left[ \omega_t \left( 1 - \frac{\gamma}{\gamma_1} \right) \frac{\delta}{\sigma_{\gamma_1,t}} + (1 - \omega_t) \left( 1 - \frac{\gamma}{\gamma_2} \right) \frac{\delta}{\sigma_{\gamma_2,t}} \right] \phi^Y \\
+ \frac{1}{\sqrt{\sigma_t \tau}} [\omega_t \Phi_{1,t} + (1 - \omega_t) \Phi_{2,t}] 
\]

(31)

and their optimal wealth process is given by:

\[
\hat{W}_t = f_{1,t} + f_{2,t},
\]

(32)

where, for \( i = 1, 2, \):

\[
f_{i,t} \equiv \left( \frac{1 + \alpha_i}{\lambda_M \xi_t} \right)^{\frac{1}{\gamma_i}} \xi_t \left( \frac{\delta}{\lambda} \right)^{1 - \frac{1}{\gamma_i}} \sqrt{\left( \frac{\delta^2 + v_t \tau'}{\gamma_i} \right) \beta_t} \left( 1 - \frac{\gamma}{\gamma_i} \right) \left( 1 - \frac{\phi^Y}{\gamma} \right) r - \left( 1 - \frac{1}{\gamma_i} \right) \left( k_i \left( \rho - ((\gamma_i - 1)k_i + 1) \frac{\delta^2}{2} \right) + \frac{1}{2\gamma_i} \frac{(\rho - \hat{\rho}_t - (\gamma_i - 1)k_i \delta^2)}{\delta^2 + \frac{v_t \tau'}{\gamma_i}} \right) \right], \Pi_{i,t}, \]

(33)

\[
\Phi_{i,t} \equiv \frac{1}{\sigma_{\gamma_i,t}} \frac{N'(d_{i,t}) - N'(\bar{d}_{i,t})}{\Pi_{i,t}},
\]

(34)

\[
d_{i,t} \equiv \gamma \delta^2 \sigma_{\gamma_i,t} \sqrt{\frac{\delta^2 + v_t \tau'}{\gamma_i}} \left( \phi_{\gamma_i,t}^N - \phi^Y \right) - \frac{\sigma_t}{\nu_t \sqrt{\tau'}} \left( \frac{\delta^2 + v_t \tau'}{\delta^2 + v_t \tau'} \right) \Gamma, \quad \bar{d}_{i,t} \equiv d_{i,t} + 2 \frac{\sigma_t}{\nu_t \sqrt{\tau'}} \left( \frac{\delta^2 + v_t \tau'}{\delta^2 + v_t \tau'} \right) \Gamma,
\]

(35)

\( \lambda_M \) is the Lagrange multiplier that solves \( \hat{W}_0 = w_0 \), \( N(.) \) is the standard normal cumulative distribution function, \( \Pi_{1,t} \equiv N(d_{1,t}) + 1 - N(\bar{d}_{1,t}), \Pi_{2,t} \equiv N(\bar{d}_{2,t}) - N(d_{2,t}), \omega_t \equiv \frac{f_{i,t}}{\hat{W}_t}, 0 \leq \omega_t \leq 1, \sigma_{\gamma_i,t} \equiv \delta + \frac{v_t \tau'}{\gamma_i}, k_i = \gamma - \frac{\gamma_i - 1}{\gamma_i} \phi^Y > 0 \) (\( i = 1, 2 \)), and \( \Gamma \geq 0 \) is as given in Appendix A.

Managers’ optimal portfolio (31) consists of the sum of three components:

1. A mean-variance component \( \omega_t \phi_{\gamma_1,t}^N + (1 - \omega_t) \phi_{\gamma_2,t}^N \). This component equals the weighted average of the normal portfolios \( \phi_{\gamma_1,t}^N \) and \( \phi_{\gamma_2,t}^N \), corresponding to RRA coef-
ponents $\gamma_1$ and $\gamma_2$, respectively. This component is proportional to, and thus has the same sign as, the normal policy $\phi_{N_{\gamma,t}}$.

2. An *indexing* component, proportional to the benchmark weight in the stock $\phi^Y$. Since we examine long-only benchmarks ($0 \leq \phi^Y \leq 1$), and the factor multiplying $\phi^Y$ in equation (31) takes values in the interval $(0, 1)$, this component adopts either long or zero positions in both the stock and the risk-free asset at all times.

3. An additional component, proportional to the sum $\omega_t \Phi_{1,t} + (1-\omega_t)\Phi_{2,t}$. Portfolios $\Phi_{1,t}$ and $\Phi_{2,t}$ are non-linear functions of the difference between the normal and benchmark portfolios $\phi_{N_{\gamma,t}} - \phi^Y$ and can reflect large long or short positions in the stock. We refer to this as a *risk-shifting* component.

We can interpret the difference between managers’ portfolio (31) and their mean-variance component as their *hedging demand* against changes in the investment opportunity set. In the current setup, these changes are given mainly by the fluctuations in the fees they expect to collect as a function of their relative performance at the end of the investment period. Thus, managers in our model hedge against the risk of underperforming or, equivalently, of not perceiving the “bonus payment” given by the performance fee in the outperformance region.

Given the decomposition above, managers’ hedging demand is captured by the indexing and risk-shifting components. This hedging demand adopts different forms within each component depending on the weight $\omega_t$. In order to interpret $\omega_t$, we first note that $\Pi_{1,t}$ and $\Pi_{2,t}$ in Proposition 3 reflect a manager’s time-$t$ conditional probability of under- and outperforming by the investment horizon $T'$, respectively. Accordingly, $f_1$ and $f_2$ in equation (33) represent the fraction of current AUM that can be attributed to managers under- and outperformance, which implies that the weight $\omega_t$ is a monotonically increasing function of the conditional probability of underperforming by $T'$. Thus, whether managers tilt their portfolio towards one sub-portfolio or the other within each component of (31) depends on their assessed probability of *underperforming* by the end of the period.

Within the indexing component, we see that managers tilt their portfolios towards the sub-portfolio $(1-\gamma/\gamma_2)\delta/\sigma_{\gamma_2,t}\phi^Y$ as the probability $1-\omega_t$ of outperforming increases. Since $0 \leq (1-\gamma/\gamma_1)\delta/\sigma_{\gamma_1,t}\phi^Y < (1-\gamma/\gamma_2)\delta/\sigma_{\gamma_2,t}\phi^Y \leq \phi^Y$, outperforming managers invest more like the benchmark than their underperforming peers. This behavior reflects the well-known
lock-in effect according to which “winning” but risk-averse managers prefer to secure an interim relative gain by investing like the benchmark against which their relative performance is assessed. In our setup, this behavior intensifies with the fee-performance sensitivity $\alpha_2$, and results in managers over-weighting an overvalued stock whenever the normal weight in the stock $\phi^N_{\gamma,t}$ is smaller than the weight implied by the indexing component of their portfolio. This situation is evidently more likely to arise for benchmarks representing high stock holdings, like those employed by most all-equity mutual funds in practice. Importantly, the indexing component leads informed managers to trade less aggressively than normal—as indicated by the normal policy—against both under- and overpricing when the benchmark is the market portfolio, $\phi^Y = \phi^U$. We expand on this point on our analysis of informed mutual funds below.

Turning to the risk-shifting portfolio, we assess the direction in which this component deviates managers’ trading from the normal policy in the following:

**Corollary 3.** At any interim state of the economy as of time $t \in [0, T')$, the sign of the risk-shifting component in manager’s portfolio (31) equals the sign of $(\phi^N_{\gamma,t} - \phi^Y)$:

$$\text{sgn} \left( \frac{1}{\sqrt{\sigma_t \tau'}} (\omega_t \Phi_{1,t} + (1 - \omega_t) \Phi_{2,t}) \right) = \text{sgn}(\phi^N_{\gamma,t} - \phi^Y).$$

(36)

Thus, whether the risk-shifting component represents a long or a short position in the stock depends on the sign of the difference in the allocations to the stock of the normal policy and the benchmark portfolio. As first pointed out by Basak, Pavlova, and Shapiro (2007), the direction in which managers shift risk is given by the risk exposure of this benchmark relative to that of the normal portfolio. In contrast with these authors’ result, though, managers’ normal policy in our setup is not constant but dependent on the (state-dependent) extent of stock mispricing as indicated by equation (30).

This result has an important implication for informed managers’ trading against mispricing: whenever $\text{sgn}(\phi^N_{\gamma,t} - \phi^Y)$ differs from $\text{sgn}(\phi^N_{\gamma,t} - \phi^U)$, managers can shift risk by over-weighting (under-weighting), relative to the normal policy, overvalued (undervalued) securities. In particular, when $\phi^Y < \phi^N_{\gamma,t} < \phi^U$, the risk-shifting component of managers’ hedging demand can lead them to hold more of an overvalued stock than the normal portfolio.\(^{18}\) We argued above that hedge funds’ benchmarks in practice can differ substantially

\(^{18}\) This can be seen by reference to equations (30) and (36). According to (30), an overvalued stock
from the market portfolio. Therefore, we expect the risk-shifting portfolio to specially distort informed hedge fund managers’ trading against mispricing. We verify this intuition in our analysis below.

Hedge Fund and Mutual Fund Managers’ Trading Against Mispricing: We now turn our attention to how the asset price dynamics described by Proposition 1 affect the trading of hedge fund and mutual fund managers’ in Proposition 3 for a typical parameterization of our model. We show that stock mispricing can potentially induce large deviations in managers’ policies from the normal trading that is to be expected under efficient financial markets.

Figure 2 illustrates the allocation in the risky asset of hedge fund (Panel 2(a)) and mutual fund (Panel 2(b)) informed managers (solid blue line) in our model, along with the allocation of the normal (dashed red line) and market (dash-and-dot black line) portfolios, in response to contemporaneous stock overvaluation \( OV_t \) as of \( t = (3/4)T' \). Managers’ depicted policies respond to the fee rates we detail in Section 2.3, with \( \phi^Y = 0 \) and \( \phi^Y = 1 \) for hedge funds and mutual funds, respectively. These benchmarks are meant to reflect the “absolute return” goal of hedge fund strategies, and the all-equity investment objective that is popular among mutual funds. The figures assume an initial stock overvaluation of 4%, corresponding to a realized dividend growth rate \( \rho \) one standard deviation lower than \( U \)-investors’ prior belief \( \rho_0 \). Any values to the right of the solid vertical line at zero correspond to situations of stock overpricing, and conversely (stock underpricing) to the left of this line. For visual reference, the figures include a vertical dotted line that marks the assumed initial overpricing, as well as a shaded area representing the true probability of the different interim overvaluation states.

We see that the trading strategies of both hedge fund and mutual fund informed managers are highly non-linear in the extent of overpricing, including situations of both higher- and lower-than-normal exposure to the stock. In partial agreement with the argument sug-

\((OV_t > 0)\) calls for a reduction in the normal stock holding, \( \phi^N_{\gamma,t} \), relative to the market’s, \( \phi^U \). According to equation (36), though, the risk-shifting component represents a long position in the stock. Under standard parameterizations of the model, this long position can lead to a long overall position \( \hat{\phi}_t \) that exceeds \( \phi^U \) (and thus \( \phi^N_{\gamma,t} \)) by a large margin.

19 The rest of the parameters of the fee rate \( f_T \) in the figure are set to capture a typical fee structure \((\alpha_1 = 0, \alpha_2 = 5, r + h = .05)\) in the hedge fund industry, or a typical F-PR \((\alpha_1 = 0, \alpha_2 = 1.5, \bar{\zeta} = .94)\) as parameterized by previous authors for mutual funds (see, e.g., Basak, Pavlova, and Shapiro (2007)).
The solid blue, dashed red and dash-and-dot black lines represent the informed managers’ \( \hat{\phi}_t \), normal \( \phi_{N,t} \) and market \( \phi^U \) portfolio weights, respectively, in the mispriced stock for different degrees of overvaluation \( OV_t \) as of \( t = (3/4)T' \). The gray area depicts the time-\( t \) conditional probability of the corresponding overvaluation state. Results correspond to a time-0 overvaluation of 4\% (corresponding to a realized dividend growth rate 1 std. dev. below the prior \( \rho_0 \)), as marked by the vertical dotted line. For hedge funds, we assume: \( \alpha_1 = 0, \alpha_2 = 5, \bar{\zeta} = 1.05, \phi^Y = 0 \). For mutual funds, we set: \( \alpha_1 = 0, \alpha_2 = 1.5, \bar{\zeta} = 0.94, \phi^Y = 1 \). The rest of the parameters are as follows: \( T' = 1, r = 1.5\%, \delta = .0128, v_0 = 0.05^2, \gamma = 5 \).

Suggested by prior literature, convex incentives can lead to aggressive trading by the managers against overvaluation that translates into a lower-than-normal weight in the stock for a moderate increase in the initial overpricing (0.05 < \( OV_t \) < 0.09, approximately).

However, this behavior is non-monotonic, and both smaller and larger stock overvaluation can lead managers to trade less aggressively than normal against the mispriced security. Moreover, for the particular realization of initial overpricing depicted in Figure 2 it can be shown that both types of managers are more likely to trade less aggressively, rather than more aggressively, against mispricing than a normal investor. Clearly, such a behavior goes against the above-mentioned argument in favor of incentive contracts in portfolio management.

The argument for incentive contracts can be particularly flawed in the case of hedge funds managers. According to Figure 2(a), informed hedge fund managers can over-weigh the overpriced stock by a large margin. These managers’ position in the overpriced stock after three quarters can not only be higher than the normal portfolio, but can also be as much as 50\% higher than the market portfolio for a level of overpricing similar to the initial
overpricing \((OV_t\text{ in a neighborhood of } 0.04)\).

This over-exposure to overpriced assets can be understood from our decomposition of managers’ portfolio as the sum of a mean-variance and a hedging components. Since the normal component behaves as expected from market efficiency, any trading implying a less-aggressive-than-normal stance against mispricing has to stem from the hedging component. Moreover, hedge funds’ absolute performance condition is equivalent to a (scaled) money market benchmark for which \(\phi^Y = 0\). This implies that the indexing component in their portfolio is invested in the risk-free asset only. Thus, hedge fund managers’ over-investment in the overvalued stock in excess of the market portfolio is due to their risk-shifting component. Crucially, the hedging demand against missing the performance fee for conditionally underperforming hedge fund managers leads them to adopt an aggressive opposite stance against mispricing relative to the normal portfolio. The intuition is that, as managers trail the hurdle performance necessary to “cash the bonus”, they optimally shift risk substantially in order to catch up with their benchmark. The direction in which managers shift risk is given by Corollary 3. Hedge funds’ benchmark entails a lower risk exposure than preferred by less-than-infinitely risk-averse managers under moderate overvaluation.\(^{20}\) Thus, they shift risk by aggressively over-weighting the overpriced stock in their portfolios, even beyond the market in some situations.\(^{21}\)

This investment pattern by the informed managers in our model is consistent with the “bubble-riding” behavior documented empirically by Brunnermeier and Nagel (2004) and Griffin, Harris, Shu, and Topaloglu (2011) for hedge funds during the build-up of the tech bubble in the late 1990s. The former authors show that several hedge funds over-weighed, relative to the market, highly overpriced technology stocks in their portfolios before the bubble burst. The latter authors find that on average hedge funds over-invested more aggressively than mutual funds on tech stocks, a situation that is highly plausible in our model according to a comparison of panels 2(a) and 2(b).

\(^{20}\) More precisely, this situation arises whenever \(\ln(1 + OV_t) + 0.5v_t\tau' < \gamma\delta^2 + v_t\tau'\) according to equation (30).

\(^{21}\) We expect this intuition to survive a multi-asset extension of our setup, where the overpriced security is just one out of \(N\) risky assets in which the manager can trade. The reason is that, as long as the overvalued asset has a positive risk premium and provides some diversification value, the normal portfolio will include a positive holding in this asset. Then, a manager levering up his normal portfolio component may indirectly lever up the overpriced asset as well. The exact extent to which the manager levers up this asset relative to all others is left for future research.
We also note that, as overpricing increases in Fig. 2(a) the hedge fund managers’ portfolio implies an exposure to the overpriced stock that is half-way between the normal and the benchmark ($\phi^Y = 0$) portfolios. This behavior is mainly driven by the indexing component in these managers’ portfolio or, equivalently, by their desire to lock-in the interim outperformance that ensures them the performance fee payment. Hence, hedge fund managers’ concerns with respect to a money market benchmark comes into play once they outperform their hurdle rate (or HWM), and leads them to over-weigh the overpriced stock in our model. This benchmark-induced conservative behavior contrasts with the common view of hedge funds as absolute-return investment vehicles.

Importantly, benchmarking concerns can induce hedge fund managers to short-sell much less of the mispriced stock than would be efficient—as dictated by the normal portfolio—for large levels of overpricing. Convex incentives in our model then lead to “self-imposed” short-sale restrictions even in the absence of explicit portfolio constraints, in agreement with the decline in short interest in NASDAQ stock during the tech bubble documented by Stein and Lamont (2004). Arguably, the incentive-based limits to short-selling that we suggest could hamper the role of sophisticated investors in stabilizing the stock market in the same fashion that explicit short-sale constraints limit pessimistic investors’ trading against overvaluation in models of disagreement (see, e.g., Hong and Stein (2007)).

Panel 2(b) shows that flow-concerned mutual fund managers can also over-invest (under-invest) in overpriced (underpriced) stocks relative to the normal portfolio. Since we set mutual funds’ benchmark equal to the market portfolio, $\phi^Y = \phi^U$, we know from Corollary 3 that the risk-shifting component in their portfolio always represents a more aggressive stance against mispricing than the normal policy. This implies that their overall less-aggressive-than-normal trading against mispricing is explained by the indexing component of their hedging demand. From our analysis above, this component induces managers to invest like the benchmark as they outperform it. Stock overvaluation in the model can then increase the probability that the benchmark is overpriced and that informed mutual fund managers outperform simultaneously, subsequently leading managers to invest more heavily in the overpriced stock. We will see in the next section that the higher the information advantage of managers over the $U$-investors, the higher the probability that managers reach an interim outperformance state and that, concerned by their benchmarks, they trade too conservatively against mispricing.

We highlight that in our analysis so far managers perceive their fees at the same time $T'$
as the stock price converges to its fundamental value. Therefore, our results do not hinge on managers’ conservatism in anticipation of potential losses triggered by further widening in mispricing, as originally suggested by Shleifer and Vishny (1997).

**Two-Period Investment Problem** (*T < T′*): In this case, managers care about their compensation in the second period. This compensation depends not only on their performance during the second period, but also on their performance in the first period.

From the timeline in Figure 1 and our description of compensation fees in Section 2.3, we can think of the two-period problem as a two-step process. From *t = 0* to *t = T*, managers’ compensation is affected by time-*T* fees *f_T*. From *t = T* to *t = T′*, the managers start off with initial wealth *f_TW_T* and receive no extra fees or income until the end of the period *T′*. We assume no intermediate consumption throughout the investment period.

We solve the managers’ investment problem recursively. First, we solve for the managers’ optimal investment, AUM and indirect utility function during the period *[T, T′]*. Second, we solve for the managers’ problem during the period *[0, T]* as a maximization of the indirect utility at *t = T*.

During the period *[T, T′]*, managers are paid a proportion of terminal AUM, which are not affected by performance fees (hedge funds) or investors’ flows (mutual funds). In this case, the incentives of the managers are the same as their delegating investors’, i.e. managers invest as if they were trading on their own account. Corollary 2 characterizes the solution to problem of the managers during *[T, T′]*:

**Corollary 4.** For *t ∈ [T, T′]*, the optimal trading strategy *ϕ_t* and wealth *W_t* of informed managers are given by:

\[
\hat{ϕ}_t = ϕ_{γ,t}^N, \tag{37}
\]

\[
\hat{W}_t = \left(λ′_Mπ_t\right)^{−\frac{1}{γ}} Z_{1−\frac{1}{γ},t,T′}, \tag{38}
\]

where \(λ′_M = π_T^{γ−1} \left(\frac{Z_{1−\frac{1}{γ},t,T′}}{w_T}\right)^{γ}\), and \(w_T\) are the AUM as of \(T\).

As expected, managers’ policies during the second period *[T, T′]* coincides with the normal policy, while their AUM differ from the normal portfolio value process (26) only by a constant.

We next characterize the optimal policy during the first period *[0, T]*. For brevity of exposition, we present only the managers’ optimal time-*T* wealth profile in the following:
Proposition 4. When \( T < T' \), the informed managers’ optimal time-\( T \) AUM are given by:

\[
\hat{W}_T = \begin{cases} 
(1 + \alpha_1)\frac{1}{\gamma} Z^{-1/\gamma} T^{-\gamma/\gamma} (\zeta Y_T) \frac{|\lambda_M \pi_T|^{-1/\gamma}}{\gamma}, & \text{if } \lambda_M \pi_T > b \left( \zeta Y_T / Z^{-1/\gamma} T^{-\gamma/\gamma} \right), \\
(1 + \alpha_2)\frac{1}{\gamma} Z^{-1/\gamma} T^{-\gamma/\gamma} (\zeta Y_T) \frac{|\lambda_M \pi_T|^{-1/\gamma}}{\gamma}, & \text{if } \lambda_M \pi_T \leq b \left( \zeta Y_T / Z^{-1/\gamma} T^{-\gamma/\gamma} \right),
\end{cases}
\]

where \( \zeta \equiv \tilde{\zeta} W_0 / Y_0 \), \( \lambda_M \) is the Lagrange multiplier that solves \( E_0[\pi_T \hat{W}_T] = w_0 \) and the function \( b(.) \) is as given in Appendix A.

We use informed managers’ optimal wealth profile (39) in the next section to gain insight about their average risk exposure to a mispriced stock when they care about performance in more than one period.

4 Numerical Analysis

In this section we use the results in propositions 3 and 4 to simulate the behavior of informed money managers under different mispricing scenarios. Our goal is to generalize the results in the previous section, with focus on the impact of the two parameters driving mispricing in our model: \( U \)-investors initial estimation error \( \rho_0 - \rho \), which controls the degree of initial overpricing, and managers’ degree of information advantage over \( U \)-investors \( v_0 \), which drives the extent of the expected mispricing in the economy.

Our baseline model parameterization assumes that managers receive their compensation fees at the end of the year, i.e. \( T = 1 \). All agents’ RRA coefficient is set to \( \gamma = 5 \). Similar to Brennan and Xia (2001) we assume a prior dividend growth rate \( \rho_0 = 0.0238 \), with associated standard error \( \sqrt{\nu_0} = 0.037 \). The dividend volatility is set to \( \delta = 0.129 \), whereas the risk-free rate is set to \( r = 0.015 \).

Our parameterization of the fee rate \( f_T \) is chosen to reflect either the typical fee structure in the hedge fund industry, or typical flow-performance relationships in the mutual fund industry. In the case of mutual funds, we consider the following parameterization: \( \alpha_1 = 0, \alpha_2 = 5, \phi^Y = 0, r + h = .05 \). In the case of mutual funds, we consider three alternative parameterizations that account for theoretically and empirically motivated differences in the sensitivity of mutual fund flows to bottom and middle-range performance. 

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22 See, e.g., Berk and Green (2004) and Huang, Wei, and Yan (2007) for models identifying mutual fund
scenarios, we assume a moderately high flow sensitivity to top performance, \( \alpha_2 = 1.5 \), and that \( I \)-investors evaluate mutual fund performance relative to the market, i.e. \( \phi^Y = 1 \). The scenarios distinguish between:

(i) No sensitivity of mutual fund flows to bottom performance, little sensitivity to medium performance: \( \alpha_1 = 0, \zeta = 1 \)

(ii) No sensitivity of mutual fund flows to bottom performance, high sensitivity to medium performance: \( \alpha_1 = 0, \zeta = .94 \)

(iii) Moderate sensitivity of mutual fund flows to bottom performance, higher sensitivity to medium performance: \( \alpha_1 = 0.5, \zeta = 1 \)

4.1 Within-Period Trading in Response to Initial Mispricing

Our analysis of the trading policies of informed managers in Section 3.2 focused on a particular realization of the dividend growth rate \( \rho \) at the beginning of the period. Figure 3 plots informed managers’ average mid-year over-investment \( \hat{\phi}_t - \phi^N_t \) in the stock, for different levels of initial overpricing \( OV_0 \). The average is computed over the cross-section of overpricing states as of \( t = 0.5 \), assuming the moderate degree of uncertainty \( \sqrt{v_0} = 0.037 \) of the baseline parameterization.

In general, for a high enough extent of mispricing all types of money managers overweigh an initially overpriced stock after half a year, and conversely for underpriced stocks. Thus, the incentives to trade against a given initial mispricing are weakest for highest under- and overvaluation. The intuition is that, the highest the initial mispricing, the more likely managers will outperform early on during the year and lock-in their outperformance from then on. Since the optimal strategy for locking-in their up-to-date performance is to partially replicate their benchmarks, informed managers will end up investing more (less) in the overpriced (underpriced) stock than the normal policy. This result highlights the potential drawbacks of using short-term incentive contracts to encourage money managers to trade against mispricing, and generalizes our observations in Section 3.2.

Once more, the case in favor of convex incentives in the context of mispricing is particularly weak for hedge fund managers (solid line): on average, these managers’ mid-year characteristics driving differences in the flow relationship. The latter authors provide empirical support for their theoretical predictions.
Managers’ average optimal weight in the stock in excess of the normal portfolio, $\hat{\phi}_t - \phi_t^N$, as of mid-year ($t = 0.5$) for different initial levels of overvaluation $OV_0$. The average is computed over the cross-section of overpricing states as of $t = 0.5$. The solid, dashed, dash-and-dot and dotted lines correspond, respectively, to the policies of a hedge fund manager and of mutual fund managers subject to the flow relationships (i) to (iii) in Section 4. Results correspond to the baseline parameterization: $\tilde{\rho}_0 = 0.0238$, $\sqrt{v_0} = 0.037$, $T = T' = 1$, $r = 1.5\%$, $\delta = 0.0129$, $\gamma = 5$. Weight in an underpriced stock will be lower than optimal. For overpriced stocks, hedge fund managers’ portfolio weight is higher than normal for both relatively low and high initial overpricing. Mutual fund managers in general still trade less aggressively than normal after half a year under medium to large levels of both under- and overpricing, but more aggressively than normal under low mispricing. Moreover, they trade less aggressively against mispricing for all levels of mispricing as the sensitivity of flows to medium and bottom performance increases. This result makes clear how investors’ flows can induce excessive trading in overpriced assets as found empirically by Brunnermeier and Nagel (2004) and by Greenwood and Nagel (2009).

Next, we look at how far the trading against mispricing of informed managers is from the normal portfolio during the whole investment period $[0, T]$. In order to focus on a meaningful distance measure, we draw on the following observations. First, we know from our analysis of Section 3.2 that mutual fund managers over- or under-weigh the stock asset
in their portfolios relative to the market only if the normal portfolio does. This implies that $(\hat{\phi} - 1)$ and $(\phi^N - 1)$ always have the same sign for mutual fund managers: positive in general for undervalued securities, negative for overvalued ones. However, how much managers over- or under-weigh the stock will differ according to the corresponding overvaluation state: if $|\hat{\phi} - 1| < |\phi^N - 1|$ (respectively, $|\hat{\phi} - 1| > |\phi^N - 1|$), we say that managers trade less (more) aggressively than normal against mispricing. Second, the tracking error volatility relative to the market (simply “tracking error” henceforth) of a portfolio with value process $W$ and weight process $\phi$ in the stock over the period $[0, T]$, defined as

$$TE \equiv StdDev(\log(R_T) - \log(R^N_T)),$$

(40)

where $R_T \equiv W_T/W_0$, is monotonically increasing in the distance $|\phi - 1|$.

This suggests associating negative (positive) values of the excess tracking error $\hat{TE} - TE^N$ of managers over the normal policy with less-aggressive-than-normal (more-aggressive-than-normal) trading against mispricing.

Third, for hedge funds (Panel 2(a)) we note that whenever $(\hat{\phi} - 1)$ and $(\phi^N - 1)$ have opposite signs managers favor mispricing, in the sense that they either invest less than the market in an underpriced stock or more than the market in an overpriced one. In these situations, hedge fund managers can take positions such that $|\hat{\phi} - 1| > |\phi^N - 1|$, implying $\hat{TE} - TE^N > 0$, even though they are effectively trading less aggressively against mispricing not only than the normal portfolio, but also than the market. As a consequence, for the hedge fund managers in our model the tracking error provides a lower bound on (i.e., overestimates) the extent of trading against mispricing.

Figure 4 plots informed managers’ average excess tracking error for different levels of initial mispricing, for both the cases $T = T'$ and $T < T'$ in Section 3.2. Clearly, the conservative stance against mispricing shown by the interim portfolios in Fig. 3 holds on average during the entire first investment period, as shown by the negative values of $\hat{TE} - TE^N < 0$ for medium to large mispricing in Panel 4(a). Moreover, this behavior remains similar even when managers have concerns with respect to a second investment period $[T, T']$, as shown by the trading pattern depicted in Panel 4(b).

Summing up, the results in this subsection suggest that informed money managers

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23 Indeed, for the wealth process $W$ in Eq. (4), we have: $TE_t = StdDev_t[\ln(W_t/S_t)] = \sigma_t|\phi_t - 1|dt$. 

32
Managers’ average excess tracking error $\hat{T}E - TE^N$ (in %) over $[0, T]$ for different initial levels of overvaluation $OV_0$. The average is computed over the cross-section of overpricing states as of $t = T$. The solid, dashed, dash-and-dot and dotted lines correspond, respectively, to the policies of a hedge fund manager and of mutual fund managers subject to the flow relationships (i) to (iii) in Section 4. Panel 4(a) shows results for the case $T = T'$, whereas Panel 4(b) shows results for the case $T = T''$. All results correspond to the baseline parameterization: $\tilde{\rho}_0 = 0.0238, \sqrt{\tilde{\nu}_0} = 0.037, T = 1, r = 1.5\%, \delta = .0129, \gamma = 5$.

Figure 4: Managers’ Excess Tracking Error in Response to Initial Mispricing

Managers responding to convex incentives can trade less aggressively against mispricing than would be consistent with market efficiency. This problem is particularly severe for sophisticated investors like hedge funds, and for both hedge fund and mutual fund managers in situations of very high and low mispricing. In the next subsection we show that this is also the case in circumstances of heightened uncertainty, more prone to large overvaluations, even though managers’ potential profits from trading against mispricing—as their information advantage over uninformed traders in the economy widens—increase.

4.2 Trading on Superior Information

In our analysis so far, we have kept the information advantage of money managers over the uninformed traders constant. Within our setup, a higher information advantage for managers results from higher uncertainty about fundamentals for the $U$-investors. Intuitively, the higher this uncertainty the more the informed traders will expect the stock to be over- or undervalued ex-ante, and the larger the profitable opportunities available to them from trading against this mispricing.
An important question we can address in our setup is then whether an increase in expected mispricing leads informed investors to trade more aggressively against it. Similar to our analysis above, we are interested in assessing whether the normal policy conforms with the standard finance paradigm of efficient markets as expected mispricing exacerbates, and how informed managers subject to convex incentives trade relative to this normal policies in the same circumstances.

We examine this problem by carrying out a comparative statics analysis based on the information advantage parameter $v_t$ that measures $U$-investors’ degree of uncertainty about $\rho$. Without loss of generality, we focus on the case $t = 0$, making the analysis ex-ante in nature: we assess informed managers’ expected trading by averaging the model results corresponding to different realizations of $\rho$ over its distribution at $t = 0$, $N(\rho_0, v_0)$. Since we assume that uninformed investors’ prior about $\rho$ is correct ($\hat{\rho}_0 = \rho_0$), $\rho_0$ can be set to equal any arbitrary value. The only remaining parameter characterizing managers’ information advantage is thus $v_0$.

Let $E\rho(.)$ denote the expectation over the distribution $N(\hat{\rho}_t, v_t)$ for $\rho$, for $t \in [0, T']$. We distinguish between:

1. **Expected Mispricing (EMP)**: refers to a situation in which $E\rho(OV_0) \neq 0$, independently of whether $S_0 > S_0^{CI}$ or $S_0 < S_0^{CI}$ in expectation (i.e. both expected stock under- and over-valuation). According to Section 3.1, this corresponds to all realizations of $\rho$ such that $\rho \neq \hat{\rho}_0 - (\gamma - \frac{1}{2}) v_0 T'$.

2. **Expected Overvaluation (EOV)**: refers to a situation in which $E\rho(OV_0) > 0$. According to Section 3.1, this corresponds to all realizations of $\rho$ such that $\rho < \hat{\rho}_0 - (\gamma - \frac{1}{2}) v_0 T'$.

The following lemma characterizes the relation between $U$-investors’ parameter uncertainty $v$ and expected stock mispricing/overvaluation:

**Lemma 1.** For $t \in [0, T']$, the time-$t$ expected mispricing $EMP(t) \equiv |E\rho[(1 + OV_t)^{\tau'}] - 1|$ and expected overvaluation $EOV(t) \equiv E\rho[(1 + OV_t)^{\tau'} | OV_t \geq 0] - 1$ are given by:

\[
EMP(t) = \left| e^{-(\gamma - 1) v_t (\tau')^2} - 1 \right|, \quad (41)
\]
\[
EOV(t) = e^{-(\gamma - 1) v_t (\tau')^2} \frac{N\left(-\left(\gamma - \frac{3}{2}\right) \sqrt{v_t \tau'}\right)}{N\left(-\left(\gamma - \frac{1}{2}\right) \sqrt{v_t \tau'}\right)} - 1 > 0. \quad (42)
\]
It can be easily seen from Lemma 1 that $EMP(t)$ is monotonically increasing in $v_t$. For all the model parameterizations that we use in our analysis throughout we check that also $EOV(t)$ is monotonically increasing in $v_t$. Hence, *there is a one-to-one positive relationship between managers’ information advantage ($U$-investors’ parameter uncertainty) and both expected mispricing and expected overvaluation in our model.*

![Graph](image)

**Figure 5: Expected Normal Tracking Error**

The solid blue and dashed red lines represent managers’ expected normal tracking error $TE_N^N$ (in %) in response to expected mispricing (Panel 5(a)) and to expected overvaluation (Panel 5(b)), for the cases $T' = 1$ and $T' = 2$, respectively. Results correspond to the following parameter values: $\tilde{\rho}_0 = 0.0238, T = 1, r = 1.5\%, \delta = .0129, \gamma = 5$.

Based on our discussion in Section 3.2, we focus our analysis below on the expected tracking error over the total investment period: $ETE \equiv E^{\rho}[TE]$. Figure 5 plots the expected normal tracking error $ETE_N^N$ for different levels of information advantage of informed over uninformed traders. We see that the trading pattern implied by the normal policy agrees with the expected behavior of informed investors under the standard paradigm of market efficiency, as well as with our results in the previous section. Specifically, trading against both mispricing (Panel 5(a)) and overvaluation (Panel 5(a)) when either $T' = 1$ (blue solid line) or $T' = 2$ (red dashed line) intensifies with the information advantage of the informed normal investor or, equivalently, with the degree of expected stock mispricing or overvaluation.

For managers subject to convex incentives, we analyze their trading against mispricing and against overvaluation for different levels of information advantage $v_0$ by computing their *excess* expected tracking error with respect to the normal expected tracking error:
For exposition brevity, we present the results corresponding only to the one-period investment ($T' = 1$) in Figure 6, as results for the two-period problem ($T' = 2$) are qualitatively very similar.

Figure 6: Managers’ Excess Expected Tracking Error
Managers’ expected excess tracking error $\hat{ET}E - ETEN$ (in %) over $[0, T]$ for different initial levels of expected mispricing (Panel 6(a)) and expected overvaluation (Panel 6(b)). The solid, dashed, dash-and-dot and dotted lines correspond, respectively, to the policies of a hedge fund manager and of mutual fund managers subject to the flow relationships (i) to (iii) in Section 4. All results correspond to the baseline parameterization: $\hat{\rho}_0 = 0.0238, T = 1, r = 1.5\%, \delta = .0129, \gamma = 5$.

We make several observations from inspection of Figure 6. First, the intensity of the trading against expected mispricing and overvaluation of informed managers generally falls with the extent of the mispricing.\textsuperscript{24} The intuition is that, as managers’ information advantage over $U$-investors increases (equivalently, expected mispricing and overvaluation rise), their probability of achieving the outperformance necessary to cash their performance fees (hedge funds) or to obtain high inflows the next period (mutual funds) increases. They then tilt their portfolio towards the hedging demand that results in a more conservative average stance against mispricing. Second, as $U$-investors uncertainty peaks, resulting in high levels of expected mispricing and overvaluation, all managers eventually trade less against the mispriced stock than would be consistent with market efficiency, as given by the normal policy ($\hat{TE} - TNE < 0$).

\textsuperscript{24} The only exception is the excess expected tracking error by the informed managers of mutual funds with no flow sensitivity to bottom relative performance but high flow sensitivity to middle-range performance (case (ii) in Section 3.2), under low levels of overpricing.
We conclude that when managers face convex incentives, their optimal investment strategy can in general be inconsistent with the market efficiency notion that trading against mispricing should intensify with the extent of mispricing. Contrary to the hypothesis that option-like incentives can induce managers to bet against overpriced securities, we have shown that this type of incentives can exacerbate, rather than lessen, informed money managers’ over-investment in these securities. This incentive-induced failure to trade against mispricing can worsen precisely when expected mispricing heightens. Following Pastor and Veronesi (2009)’s interpretation of a bubble in stock prices as a period of high uncertainty about the productivity of a technology, we conclude that convex incentives can induce “bubble-riding” behavior by informed money managers even in the absence of career concerns or other market frictions.

5 Conclusions

In this paper we consider the effect of convex incentives on the trading behavior against mispricing of a money manager with superior information. According to the standard paradigm of efficient financial markets, an agent with superior information should trade depending on the level of mispricing or deviation of the security price from the fundamentals; in particular, the investor should hold less (respectively, more) of an overpriced (underpriced) security than the market portfolio when this is mostly determined by the investment decisions of less informed traders. Even when the informed agent features career concerns, short-term incentive contracts should induce him to trade in this efficient way against overpricing and offset the bubble-riding behavior resulting from these concerns.

We find that with convex incentives the standard paradigm and the rationale for incentive contracts can no longer be valid. In particular, it is possible for money manager to over-invest (relative to the efficient level or even to the market portfolio) in overpriced securities, so as if “riding the bubble.” We further show that this behavior becomes more pronounced as expected overpricing increases. Our model is able to reconcile some puzzling empirical findings without recurring to behavioral arguments, and only using incentives documented in the literature—although not standard in financial models. We do not solve a full equilibrium model, but such (optimal) behavior would arguably “fuel” the bubble. A full equilibrium analysis is left for future research.
Appendix

A Proofs and Auxiliary Results

We start by stating two auxiliary lemmas that are used throughout the remaining proofs.

Lemma A1. Let \( \tau' = T' - t \). For \( 0 \leq t \leq T' \), \( \alpha \in \mathbb{R} \):

\[
\tilde{E}_t [D^\alpha_{T'}] = D_t^\alpha \exp \left\{ \left( \alpha \tilde{\rho}_t - \frac{1 - \alpha}{2} \alpha \delta^2 + \frac{\alpha^2}{2} v_t \tau' \right) \tau' \right\} \tag{43}
\]

Proof. The dynamics of \( D_t \) under the filtered probability are \( dD_t = D_t (\tilde{\rho}_t dt + \delta d\tilde{B}_t) \), or, for \( 0 \leq t \leq t' \leq T' \):

\[
D_{t'} = D_t e^{\int_t^{t'} \left( \tilde{\rho}_s - \frac{\delta^2}{2} \right) ds + \delta (\tilde{B}_{t'} - \tilde{B}_t)}
\]

(44)

= \[D_t e^{-\frac{\delta^2}{2} (t' - t) + \int_t^{t'} \tilde{\rho}_s ds + \delta (\tilde{B}_{t'} - \tilde{B}_t)}.
\]

From (12), and using the solution to \( v \) as: \( v_t = \frac{\delta^2 v_0}{\sigma^2 + v_0 t} \),

\[
\frac{d\tilde{\rho}_t}{v_t} = \delta d\tilde{B}_t \Rightarrow \int_t^{t'} \frac{d\tilde{\rho}_t}{v_t} = \delta (\tilde{B}_{t'} - \tilde{B}_t)
\]

(45)

\[
\Rightarrow \frac{\tilde{\rho}_t}{v_t} - \frac{\tilde{\rho}_t}{v_t} - \int_t^{t'} \tilde{\rho}_s ds = \delta (\tilde{B}_{t'} - \tilde{B}_t)
\]

\[
\Rightarrow \frac{\tilde{\rho}_t}{v_t} - \frac{\tilde{\rho}_t}{v_t} = \int_t^{t'} \tilde{\rho}_s ds + \delta (\tilde{B}_{t'} - \tilde{B}_t),
\]

which allows us to re-express (44) as:

\[
D_t e^{-\frac{\delta^2}{2} (t' - t) + \int_t^{t'} \tilde{\rho}_s ds + \delta (\tilde{B}_{t'} - \tilde{B}_t)} = D_t e^{-\frac{\delta^2}{2} (t' - t) + \int_t^{t'} \tilde{\rho}_s ds + \delta (\tilde{B}_{t'} - \tilde{B}_t)}.
\]

(46)

Note that, conditioning on \( \mathcal{F}_t^D \), the only random variable in the former expression is \( \tilde{\rho}_t \). Moreover, from (12) we know that \( \tilde{\rho}_t \) is a linear diffusion with deterministic volatility, so:

\[
\tilde{\rho}_{t'} | \tilde{\rho}_t = \tilde{\rho}_t + \frac{1}{\delta} \int_t^{t'} v_s d\tilde{B}_s | \tilde{\rho}_t 
\approx N \left( \tilde{\rho}_t, \sigma^2_{\tilde{\rho},t,t'} \right),
\]

(47)

with \( \sigma^2_{\tilde{\rho},t,t'} = \frac{1}{\delta^2} \int_t^{t'} (v_s)^2 ds = v_t - v_{t'} \). This implies that \( D_{T'} | D_t \) is log-normally distributed with
deterministic mean and variance, so:

\[
\tilde{E}_t[D_{\alpha T'}] = D_t^\alpha \exp \left\{ -\alpha \frac{\delta^2}{2} \tau' - \alpha \delta^2 \left( \frac{1}{\nu_t} - \frac{1}{\nu_{T'}} \right) \tilde{\rho}_t + \frac{\alpha^2 \delta^4}{2v_{T'}^2} (v_t - v_{T'}) \right\},
\]

(48)

which results in (43) after some algebraic manipulations.

\[\square\]

**Lemma A2.** Let \( z \sim N(0, \sigma_z^2) \), and let \( \tilde{\rho}, c, \bar{z}, \tilde{z} \in \mathbb{R} \). We have:

\[
E \left[ e^{-\tilde{\rho}(z-c)^2} \mathbb{1}_{\{z \leq \bar{z}\}} \right] = e^{-\frac{\tilde{\rho}^2}{1+2\tilde{\rho}^2} \sigma_z^2} \mathcal{N} \left( \frac{\bar{z} - \frac{2\tilde{\rho}\sigma_z^2}{1+2\tilde{\rho}^2} c}{\sigma_z/\sqrt{1+2\tilde{\rho}^2}} \right),
\]

(49)

where \( \mathcal{N}(\cdot) \) is the standard normal cumulative distribution function.

**Proof.** Follows from direct integration against the normal density, using the change of variables

\[ z = \frac{\tilde{z} - \frac{2\tilde{\rho}\sigma_z^2}{1+2\tilde{\rho}^2} c}{\sigma_z/\sqrt{1+2\tilde{\rho}^2}}, \quad \tilde{z} = \frac{z - \frac{2\tilde{\rho} \sigma_z^2}{1+2\tilde{\rho}^2}}{\sigma_z/\sqrt{1+2\tilde{\rho}^2}}. \]

\[\square\]

**Proof of Proposition 1.** The standard solution to uninformed investors’ optimization problem is:

\[ W_{T'}^{U} = (\lambda \tilde{\pi}_{T'})^{-\gamma} \Rightarrow \tilde{\pi}_{T'} = \frac{1}{\lambda} (W_{T'}^{U})^{-\gamma}. \]

(50)

By market clearing condition (11):

\[ W_{T'}^{U} = D_{T'} \Rightarrow \tilde{\pi}_{T'} = \frac{1}{\lambda} D_{T'}^{-\gamma}. \]

(51)

Uninformed investors’ equilibrium SPD is:

\[ \tilde{\pi}_t = e^{r(t'-t)} \tilde{E}_t[\tilde{\pi}_{T'}] = e^{r(t'-t)} \tilde{E}_t \left[ D_{T'}^{-\gamma} \right]. \]

(52)

Applying Lemma A1 for \( \alpha = -\gamma \), uninformed investors’ equilibrium SPD is then:

\[ \tilde{\pi}_t = \lambda^{-1} D_t^{-\gamma} \exp \left\{ \left( r - \gamma \tilde{\rho}_t + \frac{1+\gamma}{2} \gamma \delta^2 + \frac{\gamma^2}{2} v_t T' \right) \tau' \right\}. \]

(53)

Using (53) to solve for \( \lambda \) in the equation \( \tilde{\pi}_0 = 1 \):

\[ \lambda = D_0^{-\gamma} \exp \left\{ \left( r - \gamma \tilde{\rho}_0 + \frac{1+\gamma}{2} \gamma \delta^2 + \frac{\gamma^2}{2} v_0 T' \right) T' \right\}. \]

(54)
By no-arbitrage, equilibrium stock prices are:

\[
S_t = \tilde{\pi}_t^{-1} \tilde{E}_t[\tilde{\pi}_{T'} D_{T'}] = (\lambda \tilde{\pi}_t)^{-1} \tilde{E}_t \left[ D_{T'}^{1-\gamma} \right]
\]  

(55)

Using Lemma A1 for \( \alpha = 1 - \gamma \) and equation (53):

\[
S_t = D_t \exp \left\{ \left( \tilde{\rho}_t - r - \gamma \delta^2 - \left( \gamma - \frac{1}{2} \right) v_t \tau' \right) \right\}
\]  

(56)

Applying Itô’s Lemma to (56)

\[
dS_t = \left( r + \gamma \left( \delta + \frac{v_t}{\delta} \tau' \right)^2 \right) S_t dt + \left( \delta + \frac{v_t}{\delta} \tau' \right) S_t d\tilde{B}_t.
\]  

(57)

Under the \( \tilde{P} \)-probability, the stock dynamics (2) can be rewritten as:

\[
dS_t = \tilde{\mu}_t S_t dt + \sigma_t S_t d\tilde{B}_t.
\]  

(58)

Comparing the drift and diffusion terms of (57) and (58) we get equations (18). \( \square \)

**Proof of Corollary 1.** Equations (19)-(20) follow by letting \( \rho_0 \to \rho \) and \( v_0 \to 0 \) in Proposition 1. To obtain (21), we divide (16) by (19) to get:

\[
\frac{S_t}{S^\ell_{T'}} = \exp \left\{ \left( \tilde{\rho}_t - \rho - \left( \gamma - \frac{1}{2} \right) v_t \tau' \right) \tau' \right\}
\]  

(59)

The result then follows from the definition of \( OV_t \). \( \square \)

**Proof of Proposition 2.** For an informed direct investor with RRA coefficient \( \tilde{\gamma} \), the normal optimization problem is:

\[
\max_{W_T'} E_0 \left[ \frac{(W_{T'})^{1-\tilde{\gamma}}}{1-\tilde{\gamma}} \right],
\]  

subject to:

\[
E_0 [\pi_{T'} W_{T'}] = w_0.
\]  

(60)

(61)

Attaching Lagrange multiplier \( \lambda_N \) to the budget constraint (61),the normal time-\( T' \) optimal wealth profile is given by the first order condition:

\[
W^{N}_{T'} = (\lambda_N \pi_{T'})^{-\frac{1}{\tilde{\gamma}}},
\]  

(62)

where the Lagrange multiplier \( \lambda_N \) is given by:

\[
\lambda_N = w_0^{-\tilde{\gamma}} \left( E_0 \left[ (\pi_{T'})^{1-\tilde{\gamma}} \right] \right)^{\tilde{\gamma}} = \left( \frac{Z_{1-\frac{1}{2},0,T'}}{w_0} \right)^{\tilde{\gamma}}.
\]  

(63)
The normal time- \( t \) (0 ≤ \( t \) ≤ \( T' \)) portfolio value \( W_t^N \) is given by the no-arbitrage condition:

\[
\pi_t W_t^N = E_t [\pi_{T'} W_{T'}^N]
\]

\[
\Rightarrow W_t^N = (\lambda_N \pi_t)^{-\frac{1}{2}} E_t \left[ \left( \frac{\pi_{T'}}{\pi_t} \right)^{1-\frac{1}{2}} \right] = (\lambda_N \pi_t)^{-\frac{1}{2}} Z_{1-\frac{1}{T'},T'},
\]

with \( Z_{1-\frac{1}{T'},T'} = E_t \left[ \left( \frac{\pi_{T'}}{\pi_t} \right)^{1-\frac{1}{2}} \right] \). The following lemma provides a closed-form expression for \( Z_{1-\frac{1}{T'},T'} \):

\[\text{Lemma A3. Let } \psi \in \mathbb{R}. \text{ For } 0 \leq t \leq t' \leq T':\]

\[
Z_{\psi,t,t'} \equiv E_t \left[ \left( \frac{\pi_{t'}}{\pi_t} \right)^{\psi} \right] = \delta^\psi \sqrt{\left( \delta^2 + v_t (t' - t) \right)} \exp \left\{ -\psi r (t' - t) - \frac{\psi (1 - \psi) \delta^2 (t' - t)}{2} \right\}
\]

\[\text{(65)}\]

\[\text{Proof. We first simplify the expression of the likelihood process } \xi_t. \text{ Defining } \kappa_t \equiv \rho - \hat{\rho}_t \delta, \text{ the likelihood process can be rewritten as:}\]

\[\xi_t = e^{-\frac{1}{2} \int_0^t \kappa_s^2 ds - \int_0^t \kappa_s dB_s}.\]

\[\text{(66)}\]

Managers see the dynamics of \( \hat{\rho} \) in (12) as:

\[d \hat{\rho}_t = \frac{v_t}{\delta} \left( \rho - \hat{\rho}_t + dB_t \right) = \frac{v_t}{\delta^2} (\rho - \hat{\rho}_t) dt + \frac{v_t}{\delta} dB_t.\]

\[\text{(67)}\]

An application of Itô’s Lemma gives the dynamics of \( \frac{\hat{\rho}_t}{v_t} \) as:

\[d \left( \frac{\hat{\rho}_t}{v_t} \right) = \frac{\rho}{\delta^2} dt + \frac{1}{\delta} dB_t\]

\[\text{(68)}\]

A further application of Itô’s Lemma to the product \( \kappa_t \frac{\hat{\rho}_t}{v_t} \) gives:

\[d \left( \kappa_t \frac{\hat{\rho}_t}{v_t} \right) = \frac{\hat{\rho}_t}{v_t} d\kappa_t + \kappa_t d \left( \frac{\hat{\rho}_t}{v_t} \right) + d \left( \kappa_t \frac{\hat{\rho}_t}{v_t} \right)
\]

\[= -\frac{\hat{\rho}_t}{\delta^2} (\kappa_t dt + dB_t) + \kappa_t \left( \frac{\rho}{\delta^2} dt + \frac{dB_t}{\delta} \right) - \frac{v_t}{\delta^3} dt
\]

\[= \frac{2}{\delta} \left( \frac{\kappa_t^2}{2} dt + \kappa_t dB_t - \frac{v_t}{2 \delta^2} dt - \frac{\rho}{2 \delta} dB_t \right)
\]

\[\Rightarrow \frac{\kappa_t^2}{2} dt + \kappa_t dB_t = \frac{\delta}{2} d \left( \kappa_t \frac{\hat{\rho}_t}{v_t} \right) + \frac{1}{2} \left( \frac{v_t}{\delta^2} dt + \frac{\rho}{\delta} dB_t \right).
\]

\[\text{(69)}\]
Integrating both side from 0 to $t$:

$$
\frac{1}{2} \int_0^t \kappa_s^2 ds + \int_0^t \kappa_s dB_s = \frac{\delta}{2} \int_0^t d \left( \kappa_s \hat{\rho}_s \right) + \frac{1}{2\delta} \int_0^t \kappa_s dB_s + \frac{\rho}{2\delta} \int_0^t dB_s. \quad (70)
$$

Direct integration gives the second integral on the RHS as: $\int_0^t \kappa_s dB_s = -\delta^2 \ln(v_t/v_0)$, which allows us to re-express the likelihood process as:

$$
\xi_t = e^{-\frac{1}{2} \int_0^t \kappa_s^2 ds - \int_0^t \kappa_s dB_s} = \sqrt{\frac{v_t}{v_0}} e^{-\frac{1}{2} \left( \frac{\kappa_s^2 + \kappa_s \rho_k}{v_t} \right) - \frac{\rho}{2\delta} \int_0^t dB_s}, \quad (71)
$$
or, for $t' \geq t$:

$$
\frac{\xi_{t'}}{\xi_t} = \frac{\delta}{\sqrt{\delta^2 + v_t(t'-t)}} e^{-\frac{1}{2} \left( (\rho-\hat{\rho}_t)^2 - (\rho-\hat{\rho}_t)^2 \right) - \frac{\rho}{2\delta} (B_{t'} - B_t)}. \quad (72)
$$

Integrating both sides of (68), from $t$ to $t'$, we can solve for $B_{t'} - B_t$ as:

$$
B_{t'} - B_t = -\frac{\rho}{\delta} (t'-t) + \delta \left( \frac{\hat{\rho}_{t'}}{v_{t'}} - \frac{\hat{\rho}_t}{v_t} \right), \quad (73)
$$
so we can re-express (72) as:

$$
\frac{\xi_{t'}}{\xi_t} = \frac{\delta}{\sqrt{\delta^2 + v_t(t'-t)}} e^{\frac{1}{2\delta^2} (\hat{\rho}_{t'}^2 - 2\rho \hat{\rho}_{t'}) - \frac{1}{2\delta^2} (\hat{\rho}_t^2 - 2\rho \hat{\rho}_t) + \frac{\rho^2}{2\delta^2} (t'-t)}. \quad (74)
$$

Given expressions (24) and (17) for managers’ and uninformed investors’ state-price deflators, and equation (74) for the likelihood process, we can write:

$$
E_t \left[ \left( \frac{\pi_{t'}}{\pi_t} \right)^\psi \right] = \delta^\psi \sqrt{\delta^2 + (1-\psi)v_t(t'-t)} \exp \left\{ -\psi \left( r + \gamma (\rho + \gamma \delta^2) + \frac{\gamma^2}{2} v_t(t'-t) - \gamma \hat{\rho}_t \right) \right\}
\times \left( t' - t \right) - \frac{\psi v_t}{2 v_t} (\hat{\rho}_t - (\rho + \gamma \delta^2))^2 \right\}
\times E_t \left[ \exp \left\{ \frac{\psi v_t}{2 \delta^2} \left( \hat{\rho}_{t'} - (\rho + \gamma \delta^2)^2 \right) \right\} \right]. \quad (75)
$$

We know from (73) that:

$$
\hat{\rho}_{t'} = \hat{\rho}_t + (\rho - \hat{\rho}_t) \frac{v_{t'}}{\delta^2} (t' - t) + \frac{v_{t'}}{\delta} (B_{t'} - B_t), \quad (76)
$$
so the under $P \hat{\rho}_{t'}$ is normally distributed with conditional mean and variance:

$$
\left\{ \begin{array}{l}
E_t[\hat{\rho}_{t'}] = \frac{\delta^2}{\delta^2 + v_t(t'-t)} \hat{\rho}_t + \frac{v_t(t'-t)}{\delta^2 + v_t(t'-t)} \rho,

\text{Var}_t[\hat{\rho}_{t'}] = \frac{\delta^2}{(\delta^2 + v_t(t'-t))^2} (t' - t).
\end{array} \right. \quad (77)
$$
We can then rewrite the expectation on the RHS of (75) as:

\[ E_t \left[ \exp \left\{ \frac{\psi}{2\psi t'} (\tilde{\rho}_t' - (\rho + \gamma \delta^2))^2 \right\} \right] = E_t \left[ \exp \left\{ \frac{\psi}{2\psi t'} (\tilde{\rho}_t' - E_t[\tilde{\rho}_t' - E_t[\tilde{\rho}_t] - E_t[\tilde{\rho}_t] + \gamma \delta^2)^2 \right\} \right]. \]  

(78)

Using Lemma A2 for \( z = \tilde{\rho}_t - E_t[\tilde{\rho}_t], \sigma_z^2 = Var_t[\tilde{\rho}_t], \tilde{\rho} = \frac{\psi}{2\psi t'}, c = \frac{\delta^2}{\delta^2 + \psi t(t' - t)}(\rho - \tilde{\rho}_t) + \gamma \delta^2, \tilde{z} = +\infty, \) we can compute this expectation as:

\[ E_t \left[ \exp \left\{ \frac{\psi}{2\psi t'} (\tilde{\rho}_t' - (\rho + \gamma \delta^2))^2 \right\} \right] = \sqrt{\frac{\delta^2 + \psi t(t' - t)}{\delta^2 + (1 - \psi)\psi t(t' - t)}} \times \exp \left\{ \frac{\psi^2 + \psi t(t' - t) \left( \frac{\delta^2}{\delta^2 + \psi t(t' - t)}(\rho - \tilde{\rho}_t) + \gamma \delta^2 \right)^2}{\delta^2 + (1 - \psi)\psi t(t' - t)} \right\}. \]  

(79)

Plugging (79) in (75) we get, after some algebraic manipulation, equation (65).

In order to derive the investment policy (25) replicating the optimal portfolio value (26), note that this can be rewritten as \( W_t^N = f(t, \eta_t), \) where the diffusion term \( \sigma_\eta \) of \( \eta \) can be computed as \( \sigma_\eta = -\psi t/\delta^2 \) and \( f \in C^{1,2}. \) Applying Itô’s Lemma the diffusion term of \( dW_t^N \) is:

\[ -\frac{\psi t}{\delta^2} \frac{\partial W_t^N}{\partial \eta_t} = W_t^N \frac{\delta^2 + \psi t' \eta_t}{\delta^2 + \psi \tau' \tilde{\gamma}} \]  

(80)

Equating (80) to the diffusion term of \( W_t \) in (4) gives the optimal portfolio (25).

\[ \square \]

**Proof of Corollary 2.** Equation (28) follows from plugging in the equilibrium values \( \eta_t = \tilde{\eta}_t + \frac{\rho - \tilde{\rho}_t}{\delta} \) and \( \tilde{\eta}_t = \gamma \sigma_t \) in equation (25), letting \( \tilde{\gamma} = \gamma, \) subtracting 1 from \( \phi_N^{-1} \) and rearranging. Since \( \delta^2 + \psi t/\gamma \tau' \) and \( \gamma \) are positive, the LHS of (28) is negative iff the numerator on the RHS is negative, i.e.:

\[ \tilde{\rho}_t > \rho + (\gamma - 1)\psi t. \]  

(81)

To obtain condition (29), we apply the natural logarithm on both sides of equation (59) to get:

\[ \ln \left( \frac{S_t}{S_t^T} \right) = -\left( \rho - \tilde{\rho}_t \right) \left( \gamma - 1 \right) \psi t' \]  

\[ \iff \frac{1}{\tau} \ln \left( \frac{S_t}{S_t^T} \right) = -\left( \rho - \tilde{\rho}_t \right) \left( \gamma - 1 \right) \psi t' \]  

\[ \iff \frac{1}{\tau} \ln \left( \frac{S_t}{S_t^T} \right) - \frac{1}{2} \psi t' = (\rho - \tilde{\rho}_t) + (\gamma - 1) \psi t'. \]  

(82)

Therefore, condition (81) holds iff condition (29) holds.

\[ \square \]
Proof of Proposition 3. Given in the second part of the proof of Proposition 4.

Proof of Corollary 4. During $[T, T']$, managers’ optimization problem is:

$$\max_{W_{T'}} E_T \left[ \frac{(W_{T'})^{1-\gamma}}{1-\gamma} \right],$$  \hspace{1cm} (83)

subject to:

$$E_T [\pi_T W_{T'}] = \pi_T f_T W_T \equiv w_T.$$  \hspace{1cm} (84)

Attaching Lagrange multiplier $\lambda'_M$ to the budget constraint (61), managers’ time-$T'$ optimal wealth profile is given by the first order condition:

$$\hat{W}_{T'} = \left(\lambda'_M \pi_{T'}\right)^{-\frac{1}{\gamma}},$$  \hspace{1cm} (85)

where the Lagrange multiplier $\lambda'_M$ is given by:

$$\lambda'_M = \pi_T^{\gamma-1} w_T^{-\gamma} \left( E_T \left[ \left( \frac{\pi_{T'}}{\pi_T} \right)^{1-\gamma} \right] \right)^{\gamma} = \pi_T^{\gamma-1} \left( \frac{Z_{1-\frac{1}{\gamma},T,T'}}{w_T} \right)^{\gamma}. \hspace{1cm} (86)$$

Managers’ time-$t$ ($T \leq t \leq T'$) optimal AUM $\hat{W}_t$ are given by the no-arbitrage condition:

$$\pi_t \hat{W}_t = E_t \left[ \pi_{T'} \hat{W}_{T'} \right]$$

$$\Rightarrow \hat{W}_t = \left(\lambda'_M \pi_t\right)^{-\frac{1}{\gamma}} E_t \left[ \left( \frac{\pi_{T'}}{\pi_t} \right)^{1-\gamma} \right] = \left(\lambda'_M \pi_t\right)^{-\frac{1}{\gamma}} Z_{1-\frac{1}{\gamma},t,T'},$$  \hspace{1cm} (87)

with $Z_{1-\frac{1}{\gamma},t,T'} = E_t \left[ \left( \frac{\pi_{T'}}{\pi_t} \right)^{1-\gamma} \right]$. The equality (37) follows from recognizing that $\hat{W}_t$ in (87) and $W_t^N$ in (64) differ only by a constant, in which case the diffusion terms in the SDEs characterizing their respective dynamics are identical.

Proof of Proposition 4. In order to characterize the optimal Policy during the first period $[0, T]$, we first need to solve for the indirect utility of informed managers at $t = T$. Let $\zeta \equiv \tilde{\zeta} W_0 / Y_0$ be the normalized performance fee threshold. We use the following

Lemma A4. As of $t = T$, managers’ indirect utility $U_T(W_T) \equiv E_T \left[ u \left( \hat{W}_{T'} \right) \right]$ (where $\hat{W}_{T'}$ is the optimal terminal AUM as given by (26) when $w_T = \pi_T f_T W_T$), is given by:

$$U_T(W_T) = \frac{Z_{1-\frac{1}{\gamma},T,T'}^{\gamma}}{1-\gamma} \times \left\{ \begin{array}{ll}
W_T^{1-\gamma_1} (\zeta Y_T)^{\gamma_1-\gamma}, & \text{if } W_T < \zeta Y_T, \\
W_T^{1-\gamma_2} (\zeta Y_T)^{\gamma_2-\gamma}, & \text{if } W_T \geq \zeta Y_T.
\end{array} \right.$$  \hspace{1cm} (88)
Proof. Using (85) we can write:

\[
(W_T')^{1-\gamma} = \left(\lambda'_M \right)^{1-\frac{1}{\gamma}} \pi_T^{1-\frac{1}{\gamma}},
\]

so:

\[
U_T(W_T) = E_T \left[u \left(W_T'\right)\right] = \left(\lambda'_M \right)^{1-\frac{1}{\gamma}} E_T \left[\frac{\pi_T^{1-\frac{1}{\gamma}}}{1-\gamma}\right].
\]  

(89)

From (86):

\[
E_T \left[\pi_T^{1-\frac{1}{\gamma}}\right] = \left(\lambda'_M \right)^{\frac{1}{\gamma}} \pi_T f_T W_T,
\]

so:

\[
U_T(W_T) = \frac{\lambda'_M \pi_T f_T W_T}{1-\gamma}.
\]  

(90)

Plugging in the value of \(\lambda'_M\) as given by (86) for \(w_T = \pi_T f_T W_T\):

\[
U_T(W_T) = \frac{(f_T W_T)^{1-\gamma} \left(E_T \left[\frac{\pi_T}{\pi_T}\right]^{1-\frac{1}{\gamma}}\right)^\gamma}{1-\gamma} = \frac{(f_T W_T)^{1-\gamma} Z_{1-\frac{1}{\gamma},T,T'}^{\gamma}}{1-\gamma}.
\]  

(91)

Finally, plugging in the expression for \(f_T\) in (6), using the definition of \(\zeta, \gamma_1\) and \(\gamma_2\) in Section 3.2, and omitting the resulting term in \(k\) (since it does not change incentives in the margin and is absorbed by the Lagrange multiplier attached to managers’ budget constraint at \(t = 0\)), we get expression (88).

At \(t = 0\), the problem of informed money managers is then:

\[
\max_{W_T} E_0 [U_T(W_T)] \quad \text{s.t.} \quad E_0[\pi_T W_T] = w_0.
\]

(94)

The objective function (88) in managers’ problem (94) is locally non-concave in a neighborhood of \(\delta W_T = \zeta Y_T\). Standard optimization techniques cannot be applied directly to this problem. Following Basak and Makarov (2014), the first step consists in constructing the concavification \(\tilde{U}_T(.)\) of managers’ indirect utility function \(U_T(.)\) (i.e. the smallest concave function \(\tilde{U}_T(w)\) satisfying \(\tilde{U}_T(w) \geq U_T(w)\) for all \(w \geq 0\)), restate and solve the original problem (94) in terms of \(\tilde{U}_T(.)\).

In order to construct the concavified function for a general intermediate horizon \(T \leq T\), we look for functions \(W\left(\zeta Y_T, Z_{1-\frac{1}{\gamma},T,T'}\right), W\left(\zeta Y_T, Z_{1-\frac{1}{\gamma},T,T'}\right), a\left(\zeta Y_T, Z_{1-\frac{1}{\gamma},T,T'}\right)\) and \(b\left(\zeta Y_T, Z_{1-\frac{1}{\gamma},T,T'}\right)\) so that (omitting the arguments for notational simplicity):

\[
\tilde{U}_T(W_T) = \begin{cases} 
U_T(W_T), & \text{if } W_T < W \leq \zeta Y_T, \\
a + bW_T, & \text{if } W \leq W_T < W, \\
U_T(W_T), & \text{if } \zeta Y_T \leq W \leq W_T,
\end{cases}
\]

(95)
and

\[ \tilde{U}'(W_T) = \begin{cases} 
U'_T(W_T), & \text{if } W_T < W \leq \zeta Y_T, \\
b, & \text{if } W \leq W_T < W, \\
U'_T(W_T), & \text{if } \zeta Y_T \leq W \leq W_T. 
\end{cases} \]  (96)

where:

\[ U'_T(W_T) = Z_{1-\frac{1}{\gamma},T,T'}^\gamma \times \begin{cases} 
(1 + \alpha_1)W_T^{-\gamma_1} (\zeta Y_T)^{\gamma_1 - \gamma}, & \text{if } W_T < \zeta Y_T, \\
(1 + \alpha_2)W_T^{-\gamma_2} (\zeta Y_T)^{\gamma_2 - \gamma}, & \text{if } W_T > \zeta Y_T.
\end{cases} \]  (97)

Eqs. (97) and using (88) give us a system of 4 equations in our 4 unknowns \( W, \bar{W}, a \) and \( b \):

\[
\begin{align*}
  a + bW &= \frac{Z_{1-\frac{1}{\gamma},T,T'}^\gamma W^{1-\gamma_1} (\zeta Y_T)^{\gamma_1 - \gamma}}{Z_{1-\frac{1}{\gamma},T,T'}^\gamma W^{1-\gamma_2} (\zeta Y_T)^{\gamma_2 - \gamma}}, \\
  a + b\bar{W} &= \frac{Z_{1-\frac{1}{\gamma},T,T'}^\gamma W^{1-\gamma_1} (\zeta Y_T)^{\gamma_1 - \gamma}}{Z_{1-\frac{1}{\gamma},T,T'}^\gamma W^{1-\gamma_2} (\zeta Y_T)^{\gamma_2 - \gamma}}, \\
  b &= (1 + \alpha_1)Z_{1-\frac{1}{\gamma},T,T'}^\gamma W^{-\gamma_1} (\zeta Y_T)^{\gamma_1 - \gamma}, \\
  b &= (1 + \alpha_2)Z_{1-\frac{1}{\gamma},T,T'}^\gamma W^{-\gamma_2} (\zeta Y_T)^{\gamma_2 - \gamma}.
\end{align*}
\]  (98)

The solution to this system of equation yields

\[ b \left( \frac{\zeta Y_T}{Z_{1-\frac{1}{\gamma},T,T'}}^\gamma \right) = \left( \frac{(1 + \alpha_2)}{(1 + \alpha_1)} \right)^{\gamma_1 - \gamma_2} \left( \frac{\gamma_2}{\gamma_1} \right)^{\gamma_1 - \gamma_2} \frac{1}{\gamma_2 - \gamma_1} \left( \frac{\zeta Y_T}{Z_{1-\frac{1}{\gamma},T,T'}}^\gamma \right)^{-\gamma}, \]  (99)

\[ W(\zeta Y_T) = \left( \frac{(1 + \alpha_2)}{(1 + \alpha_1)} \right)^{\gamma_2 - \gamma_1} \left( \frac{\gamma_2}{\gamma_1} \right)^{\gamma_2 - \gamma_1} \zeta Y_T, \]  (100)

\[ \bar{W}(\zeta Y_T) = \left( \frac{(1 + \alpha_1)}{(1 + \alpha_2)} \right)^{\gamma_1 - \gamma_2} \left( \frac{\gamma_2}{\gamma_1} \right)^{\gamma_1 - \gamma_2} \zeta Y_T. \]  (101)

In order to verify that (99) to (101) are indeed the solutions we are after, it remains to verify that \( W \) and \( \bar{W} \) satisfy the condition:

\[ W \leq \zeta Y_T \leq \bar{W}, \]  (102)

which holds iff:

\[ \left( \frac{(1 + \alpha_2)}{(1 + \alpha_1)} \right)^{\gamma_2 - \gamma_1} \left( \frac{\gamma_2}{\gamma_1} \right)^{\gamma_2 - \gamma_1} \frac{1}{\gamma_2 - \gamma_1} < 1, \]  (103)

and

\[ \left( \frac{(1 + \alpha_1)}{(1 + \alpha_2)} \right)^{\gamma_1 - \gamma_2} \left( \frac{\gamma_2}{\gamma_1} \right)^{\gamma_1 - \gamma_2} \frac{1}{\gamma_2 - \gamma_1} > 1. \]  (104)

Since

\[ \frac{1 + \alpha_2}{1 + \alpha_1} < \left( \frac{1 + \alpha_2 \gamma_1}{1 + \alpha_1 \gamma_2} \right)^{\gamma_2}, \]  (105)

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and
\[
\frac{1 + \alpha_1}{1 + \alpha_2} < \left( \frac{1 + \alpha_1 \gamma_2}{1 + \alpha_2 \gamma_1} \right)^{\gamma_1},
\]
both conditions indeed verify.

We can now restate managers’ optimization problem (94) at \( t = 0 \) as:
\[
\max_{W_T} E_0 \left[ \tilde{U}_T(W_T) \right] \quad \text{s.t.} \quad E_0[\pi_T W_T] = w_0.
\]
(107)

Attaching Lagrange multiplier \( \lambda_M \) to the budget constraint, the solution to the concavified problem (107) is given by the standard (state-by-state) first order condition:
\[
\tilde{U}_T'(W_T) = \lambda_M \pi_T.
\]
(108)

Using (97):
\[
\lambda_M \pi_T = Z_{1 - \frac{1}{\gamma}, \bar{T}, T'}^T \times \begin{cases} 
(1 + \alpha_1) W_T^{-\gamma_1} (\zeta Y_T)^{\gamma_1 - \gamma}, & \text{if } W_T < \zeta Y_T, \\
\bar{W}, & \text{if } \bar{W} \leq W_T < W, \\
(1 + \alpha_2) W_T^{-\gamma_2} (\zeta Y_T)^{\gamma_2 - \gamma}, & \text{if } W_T > \zeta Y_T,
\end{cases}
\]
which gives managers’ optimal time-\( T \) AUM as:
\[
\hat{W}_T = \begin{cases} 
(1 + \alpha_1) \frac{1}{\gamma_1} Z_{1 - \frac{1}{\gamma}, \bar{T}, T'}^T (\zeta Y_T)^{\gamma_1 - \gamma} (\lambda_M \pi_T)^{-\frac{1}{\gamma_1}}, & \text{if } \lambda_M \pi_T > b \\
W \in [\bar{W}, \bar{W}], & \text{if } \bar{W} \leq W_T < \bar{W}, \\
(1 + \alpha_2) \frac{1}{\gamma_2} Z_{1 - \frac{1}{\gamma}, \bar{T}, T'}^T (\zeta Y_T)^{\gamma_2 - \gamma} (\lambda_M \pi_T)^{-\frac{1}{\gamma_2}}, & \text{if } \bar{W} \leq W_T.
\end{cases}
\]
(110)

Using Eqs. (99) through (101), we note that:
\[
\hat{W}_T < \bar{W} \iff \lambda_M \pi_T > b,
\]
(111)
and
\[
\hat{W}_T \geq \bar{W} \iff \lambda_M \pi_T \leq b,
\]
(112)
which allows us to re-express (110) as:
\[
\hat{W}_T = \begin{cases} 
(1 + \alpha_1) \frac{1}{\gamma_1} Z_{1 - \frac{1}{\gamma}, \bar{T}, T'}^T (\zeta Y_T)^{\gamma_1 - \gamma} (\lambda_M \pi_T)^{-\frac{1}{\gamma_1}}, & \text{if } \lambda_M \pi_T > b, \\
(1 + \alpha_2) \frac{1}{\gamma_2} Z_{1 - \frac{1}{\gamma}, \bar{T}, T'}^T (\zeta Y_T)^{\gamma_2 - \gamma} (\lambda_M \pi_T)^{-\frac{1}{\gamma_2}}, & \text{if } \lambda_M \pi_T \leq b.
\end{cases}
\]
(113)

This completes our proof of Proposition 4. We now prove Proposition 3. In order to derive Eqs.
Using the above closed-form expressions for \( \pi \), we can express \( b \) which implies:

\[
\left\{ \begin{array}{ll}
(1 + \alpha_1) \frac{1}{2} \left( \gamma T_{\pi T'} \gamma \right) \frac{1}{1} \left( \lambda_{M T'} \pi \right) - \frac{1}{2}, & \text{if } \lambda_{M T'} > b (\gamma T_{T'}), \quad \mathcal{R}_1, \\
(1 + \alpha_2) \frac{1}{2} \left( \gamma T_{\pi T'} \gamma \right) \frac{1}{1} \left( \lambda_{M T'} \pi \right) - \frac{1}{2}, & \text{if } \lambda_{M T'} \leq b (\gamma T_{T'}), \quad \mathcal{R}_2.
\end{array} \right.
\]

In order to define regions \( \mathcal{R}_1 \) and \( \mathcal{R}_2 \), we need to obtain an explicit expression for the benchmark \( Y_{T'} \). This can be done more easily by first writing the dynamics of \( Y_t \) under the uninformed investors’ probability \( \tilde{P} \):

\[
dY_t = Y_t \left( r + \phi^Y \sigma_t \tilde{\eta}_t \right) dt + Y_t \phi^Y \sigma_t d\tilde{B}_t,
\]

which implies:

\[
Y_{T'} = Y_t \exp \left\{ rT' + \phi^Y \gamma \left( \gamma - \frac{\phi^Y}{2} \right) \int_t^{T'} \sigma_s^2 ds + \phi^Y \int_t^{T'} \sigma_s \tilde{d}B_s \right\}
\]

\[
= Y_t \exp \left\{ \left( r + \phi^Y \gamma \left( \gamma - \frac{\phi^Y}{2} \right) \right) (\delta^2 + v T') \phi^Y \right\} T' + \frac{\delta^2 \phi^Y}{v T'} (\tilde{\rho}_{T'} - \tilde{\mu}) \right\},
\]

where we used expressions (12) and (18) to solve for the integrals in (116). Defining:

\[
\zeta_0 \equiv \left[ \frac{(1 + \alpha_2) \gamma_1 \gamma_2 - 1}{(1 + \alpha_1) \gamma_1 \gamma_2 - 1} \right] \left( \frac{\gamma_1}{\gamma_2} \right)^{\gamma_1 \gamma_2} \gamma_2 - \gamma_1,
\]

we can express \( b (\gamma T_{T'}) = \zeta_0 (\gamma T_{T'})^{-\gamma} \). Region \( \mathcal{R}_1 \) is then given by:

\[
\lambda_{M T'} > b (\gamma T_{T'}) \iff \lambda_{M T'} > \zeta_0 (\gamma T_{T'})^{-\gamma}.
\]

Using the above closed-form expressions for \( \pi_{T'} \) and \( Y_{T'} \) we can express region \( \mathcal{R}_1 \) as:

\[
\{ \tilde{\rho}_{T'} < \rho + (1 - \phi^Y) \gamma \delta^2 + \Gamma \} \text{ or } \{ \tilde{\rho}_{T'} > \rho + (1 - \phi^Y) \gamma \delta^2 + \Gamma \},
\]

where:

\[
\Gamma \equiv \sqrt{v T'} \Delta (\tilde{\rho}_0, v_0),
\]

and

\[
\Delta (\tilde{\rho}_0, v_0) \equiv \frac{1}{v_0} \left( \tilde{\rho}_0 - \rho - (1 - \phi^Y) \gamma \delta^2 \right)^2
\]

\[
+ 2 \left( 1 - \phi^Y \right) \gamma \left\{ \ln \left( \frac{D_0}{\tilde{\beta}_0} \right) - r \left( 1 - (1 - \phi^Y) \gamma \right) \frac{\delta^2}{2} - \rho \right\} T'
\]

\[
+ 2 \ln \left( \frac{\lambda \zeta_0}{\lambda_{M T'} \delta T'} \sqrt{\delta^2 + v_0 T'} \right).
\]

The existence of a solution to managers’ problem (94) requires \( \Delta (\tilde{\rho}_0, v_0) > 0 \), implying \( \Gamma \geq 0 \). Region \( \mathcal{R}_2 \) is just the relative complement in \( \mathbb{R} \) of \( \mathcal{R}_1 \). We can now derive the interim AUM (32).
By no-arbitrage, the deflated wealth process $\pi_t \hat{W}_t$ is a martingale, so using (114) the optimal wealth $\hat{W}_t$ for all $t \in [0, T']$ is given by:

$$\pi_t \hat{W}_t = E_t \left[ \pi_{T'} \hat{W}_{T'} \right] \Rightarrow \hat{W}_t = f_{1,t} + f_{2,t},$$

where:

$$f_{i,t} = \pi_t E_t \left[ (1 + \alpha_i) \gamma_i \frac{1}{\gamma_i} \left( \zeta \gamma_i \right)^{\gamma_i} \left( \lambda M \pi_{T'} \right)^{-\gamma_i} 1_{R_1} \right].$$

Using the closed-form expressions above for $\pi_{T'}$ and $Y_{T'}$, $R_1$ and $R_2$, and applying Lemma A2 to compute the expectation in (123), we get Eq. (33) and thus managers’ optimal wealth process (32). The Lagrange multiplier $\lambda_M$ is the solution to the equation $\hat{W}_0 = w_0$.

In order to derive the investment policy (31) replicating the optimal portfolio value (32), note that this can be rewritten as $\hat{W}_t = h(t, D_t, \pi, \xi_t, X_{1,t}, X_{2,t}, d_{1,t}, d_{2,t})$, where for $i = 1, 2$:

$$X_{i,t} = \exp \left\{ \left[ \left( 1 - \frac{\gamma_i}{\gamma_i} \right) (1 - \theta Y) r - \left( 1 - \frac{1}{\gamma_i} \right) \left( k_i \left( \rho - \left( (\gamma_i - 1)k_i + 1 \right) \frac{\delta^2}{2} \right) + \frac{1}{2\gamma_i} \left( \rho - \bar{\rho}_i - (\gamma_i - 1)k_i \delta^2 \right)^2 \right) \right] t' \right\},$$

for some function $h \in C^{1,2}$. Applying Itô’s Lemma the diffusion term $\sigma_{\hat{W}}$ of $d\hat{W}_t$ is:

$$\sigma_{\hat{W}} = \sigma_D h_D + \sigma_{\bar{h}} h_{\bar{h}} + \sigma_\xi h_\xi + \sigma_X h_X + \sigma_{\bar{d}_1} h_{\bar{d}_1} + \sigma_{\bar{d}_2} h_{\bar{d}_2} + \sigma_{\bar{d}_1} h_{\bar{d}_1} + \sigma_{\bar{d}_2} h_{\bar{d}_2},$$

where $h_x$ denotes the partial derivative of $h$ w.r.t. $x$ and $\sigma_X$ is the diffusion term in the SDE characterizing the dynamics of the process $X$. Computing the diffusion terms in (125) explicitly and equating the result to the diffusion term of $W_t$ in (4) gives the optimal portfolio (31).

\[ \square \]

**Proof of Corollary 3.** From equation (34), for $i = 1, 2$ the sign of each risk-shifting component $\Phi_{i,t}$ equals the sign of $\mathcal{N}'(d_{i,t}) - \mathcal{N}'(\bar{d}_{i,t})$. By the symmetry of the standard normal density, $\mathcal{N}'(d_{i,t}) \geq \mathcal{N}'(\bar{d}_{i,t})$ if and only if $|d_{i,t}| \leq |\bar{d}_{i,t}|$. Since $d_{i,t} < \bar{d}_{i,t}$,

$$|d_{i,t}| \leq |\bar{d}_{i,t}| \Leftrightarrow d_{i,t} + \bar{d}_{i,t} \geq 0$$

$$\Leftrightarrow 2 \frac{\gamma \delta \sigma_{\gamma,t}}{\nu \sqrt{\tau'}} \sqrt{\frac{\delta^2 + \nu_1 \tau'}{\delta^2 + \nu_2 \tau'}} \left( \phi_{\gamma,t}^N - \phi^Y \right) \geq 0.$$  

(126)

Since the factor multiplying the difference $(\phi_{\gamma,t}^N - \phi^Y)$ above is positive, we conclude that $\mathcal{N}'(d_{i,t}) \geq \mathcal{N}'(\bar{d}_{i,t})$ if and only if $\phi_{\gamma,t}^N \geq \phi^Y$. Thus, for $i = 1, 2$, $\text{sgn}(\Phi_{i,t}) = \text{sgn}(\phi_{\gamma,t}^N - \phi^Y)$, which leads to equation (36).

\[ \square \]
Proof of Lemma 1. As of time $t$, $\rho \sim N(\tilde{\rho}_t, v_t)$ which implies that $S_t/S_t^{CI}$ as given by (82) is log-normally distributed. We then have:

$$E^\rho \left[ \frac{S_t}{S_t^{CI}} \right] = E^\rho \left[ e^{-((\rho - \tilde{\rho}_t) + (\gamma - \frac{1}{2}) v_t \tau'))} \right] = e^{-(\gamma - 1) v_t (\tau')^2},$$

which, given the definition of $EMP$, leads to (41).

To compute the expected overvaluation $EOV$ we first note that, by definition, $EOV > 0$. Moreover,

$$E^\rho \left[ \frac{S_t}{S_t^{CI}} \mathbbm{1}_{\{\rho < \tilde{\rho}_t - (\gamma - \frac{1}{2}) v_t \tau')\}} \right] = e^{-(\gamma - 1) v_t (\tau')^2} E^\rho \left[ e^{-((\rho - \tilde{\rho}_t) + (\gamma - \frac{1}{2}) v_t \tau'))} \mathbbm{1}_{\{\rho < \tilde{\rho}_t - (\gamma - \frac{1}{2}) v_t \tau')\}} \right].$$

Since $\rho - \tilde{\rho}_t \sim N(0, v_t)$, integrating against the normal density we can compute the expectation on the RHS of (128) as:

$$E^\rho \left[ e^{-((\rho - \tilde{\rho}_t) + (\gamma - \frac{1}{2}) v_t \tau'))} \mathbbm{1}_{\{\rho < \tilde{\rho}_t - (\gamma - \frac{1}{2}) v_t \tau')\}} \right] = e^{\frac{1}{2} v_t (\tau')^2} N \left( -\left( \gamma - \frac{3}{2} \right) \sqrt{v_t \tau'} \right).$$

Moreover,

$$Prob \left( \rho < \tilde{\rho}_t - \left( \gamma - \frac{1}{2} \right) v_t \tau' \right) = Prob \left( \frac{\rho - \tilde{\rho}_t}{\sqrt{v_t}} < - \left( \gamma - \frac{1}{2} \right) \sqrt{v_t \tau'} \right) = N \left( -\left( \gamma - \frac{1}{2} \right) \sqrt{v_t \tau'} \right).$$

Then,

$$EOV(t) = E^\rho \left[ \frac{S_t}{S_t^{CI}} \mid \rho < \tilde{\rho}_t - \left( \gamma - \frac{1}{2} \right) v_t \tau' \right] - 1 = \frac{E^\rho \left[ \frac{S_t}{S_t^{CI}} \mathbbm{1}_{\{\rho < \tilde{\rho}_t - \left( \gamma - \frac{1}{2} \right) v_t \tau')\}} \right]}{Prob \left( \rho < \tilde{\rho}_t - \left( \gamma - \frac{1}{2} \right) v_t \tau' \right)} - 1 = e^{-(\gamma - 1) v_t (\tau')^2} N \left( -\left( \gamma - \frac{3}{2} \right) \sqrt{v_t \tau'} \right) - \frac{N \left( -\left( \gamma - \frac{3}{2} \right) \sqrt{v_t \tau'} \right)}{N \left( -\left( \gamma - \frac{1}{2} \right) \sqrt{v_t \tau'} \right)} - 1. \tag{131}$$

\hfill \Box
References


